Simple birth

Model:

- Living individuals give birth independently of one another
- Each individual gives birth with probability $\lambda h + o(h)$
- No death

Claim:

The simple birth process is a birth process with $\lambda_n = n\,\lambda$

Model If $N(t) = n$, then every individual (among the n individuals) has a birth rate $= \lambda$. All indiv. are $\perp\!\!\!\perp$

Thus locally (on $[t, t+h)$),

$$N(t+h) - N(t) \sim \mathrm{Bin}(n, \lambda h)$$

Formula For small h

$$\mathbb{P}(N(t+h) - N(t) = m \mid N(t) = n)$$

$$= \underbrace{\binom{n}{m} (\lambda h)^m (1 - \lambda h)^{n-m}} + o(h)$$

– we have to discard every term which is $o(h)$

We have

$$\mathbb{P}(N(t+h)-N(t) = m \mid N(t)=n)$$

$$= \binom{n}{m} (\lambda h)^m (1-\lambda h)^{n-m} + o(h)$$

Case $m=0$

$$\mathbb{P}(N(t+h)-N(t) = 0 \mid N(t)=n)$$

$$= 1 \times (\lambda h)^0 (1-\lambda h)^n + o(h)$$

$$= 1 - n\lambda h + o(h)$$ (Taylor expansion order 1)

Case $m=1$

$$\mathbb{P}(N(t+h)-N(t) = 0 \mid N(t)=n)$$

$$= n \lambda h (1-\lambda h)^{n-1} + o(h) = n\lambda h + o(h)$$

Case $m > 1$

$$\mathbb{P}(N(t+h) - N(t) = m \mid N(t) = n)$$

$$= \binom{n}{m} \underbrace{(\lambda h)^m}_{= o(h)} (1 - \lambda h)^{n-m} + o(h)$$

$$= o(h)$$

Summary

birth process with $\lambda_n = n\lambda$

$$\mathbb{P}(N(t+h) - N(t) = m \mid N(t) = n)$$

$$= \begin{cases} 1 - n\lambda h + o(h) & \text{if } m = 0 \\ n\lambda h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \end{cases}$$

Simple birth (2)

Justification of the claim: Let $M = \#$ births in $(t, t+h)$. Then

$$\mathbf{P}\left(M = n \middle| N(t) = n\right) = \binom{n}{m} (\lambda h)^{m} (1 - \lambda h)^{n-m} + o(h)$$

$$= \begin{cases} n\lambda h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - n\lambda h + o(h) & \text{if } m = 0 \end{cases}$$

Simple birth with immigration

Model:

- Living individuals give birth independently of one another
- Each individual gives birth with probability $\lambda h + o(h)$
- No death
- Constant immigration ν

Form of λ_n: We get

$$\lambda_n = n\,\lambda + \nu$$

Recall For Poisson, we have found a formula for

$$\tilde{p}_j(t) = \mathbb{P}(N(t) = j) = P(N(t) = j \mid N(0) = 0)$$

we could have obtained

$$\boxed{p_{ij}(t)} = \mathbb{P}(N(s+t) = j \mid N(s) = i)$$

$$= \tilde{p}_{j-i}(t) \quad \boxed{= e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!}}$$

Here our aim is to repeat that for a general birth process

Forward ode's for the probabilities

Proposition 6.

Rmk $p_{ij}(t) = 0$ if $j < i$

Let

- N birth process
- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$

Set

$$p_{ij}(t) = \mathbf{P}\left(N(s+t) = j \,\middle|\, N(s) = i\right)$$

Then for $j \geq i$ the function p_{ij} satisfies

$$p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t),$$

∞ system of linear diff equations

with initial condition $p_{ij}(0) = \delta_{ij}$

Backward ode's

Proposition 7.

in order to know p_{ij}, I must know

$p_{i+1,j}$ $\longrightarrow$ backward

Let

- N birth process
- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$

Set

$$p_{ij}(t) = \mathbf{P}\left(N(s+t) = j \middle| N(s) = i\right)$$

Then for $j \geq i$ the function p_{ij} satisfies

$$p'_{i,j}(t) = \lambda_i p_{i+1,j}(t) - \lambda_i p_{i,j}(t)\,,$$

with initial condition $p_{ij}(0) = \delta_{ij}$

Solving the forward system

$p_{ij}(t) = P(N(s+t)=j \mid N(s)=i)$

$\Rightarrow p_{ij}(0) = P(N(s)=j \mid N(s)=i)$

$= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} \overset{\Delta}{=} \delta_{ij}$

notation

Theorem 8.

Let

- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$
- Set of indices $\{0 \leq i, j < \infty\}$

Then the system of equations

- $p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t)$ if $j \geq i$
- $p_{ij}(0) = \delta_{ij}$
- $p_{ij}(t) = 0$ if $j < i$

admits a unique solution

Equations $\quad p'_{ij} = \lambda_{j-1}\, p_{i,j-1} - \lambda_j\, p_{ij}$

Case $j<i$: $\quad p_{ij} = 0$

Case $j=i$: Equation becomes

$$p'_{ii} = -\lambda_{i-1}\, \overbrace{p_{i,i-1}}^{0} - \lambda_i\, p_{ii}$$

$$\begin{cases} p'_{ii} = -\lambda_i\, p_{ii} \longrightarrow \text{separable} \\ p_{ii}(0) = 1 \end{cases}$$

$$\Rightarrow \boxed{p_{ii}(t) = e^{-\lambda_i t}}$$

Equations $p'_{ij} = \lambda_{j-1} p_{i,j-1} - \lambda_j p_{ij}$

If we know $p_{i,j-1}$ for $j > i$, then

$$p'_{ij} + \lambda_j p_{ij} = \overbrace{\lambda_{j-1} p_{i,j-1}}^{\text{known}}$$

linear diff eq, with integrating factor $e^{\lambda_j t}$

Easily solvable $\Rightarrow$ $\boxed{\text{unique solution for the system}}$

Proof of Theorem 8

Case $i = j$: The equation becomes

$$p'_{i,i}(t) = -\lambda_i p_{i,i}(t), \quad \text{initial condition} \quad p_{i,i}(0) = 1$$

Thus

$$p_{i,i}(t) = \exp(-\lambda_i t)$$

General case:

Obtained by recursion