Equations $p'_{ij} = \lambda_{j-1} p_{i,j-1} - \lambda_j p_{ij}$

Of the form

"diff in time of p_{ij} = diff operator in space"

This is a $\boxed{\text{partial differential equation}}$

Poisson case

$$p'_{ij} = \lambda p_{i,j-1} - \lambda p_{ij}$$

$$p'_{ij} = -\lambda (p_{i,j} - p_{i,j-1})$$

$$\boxed{\frac{\partial p_{ij}}{\partial t} = -\lambda \nabla p_{i,j}}$$

Rmk In diff equations, an important tool is Laplace transforms

The same is true for pde's

In particular, useful for transition densities

Laplace transform

Definition: Let $f : \mathbb{R}_+ \to \mathbb{R}$. Then

$$\mathcal{L}f(s) = \hat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Possible strategy to solve a differential equation:

1. Transform diff. equation into algebraic problem in s variable.
2. Solve algebraic problem and find $\hat{f}$.
3. Invert Laplace transform and find f.

Existence of Laplace transform

Theorem 9.

Hypothesis:

- f piecewise continuous on $[0, A]$ for each $A > 0$.
- $|f(t)| \leq Ke^{at}$ for $K \geq 0$ and $a \in \mathbb{R}$.

Conclusion:
$\mathcal{L}f(s)$ exists for $s > a$.

Vocabulary: f satisfying $|f(t)| \leq Ke^{at}$
$\hookrightarrow$ Called function of exponential order.

Table of Laplace transforms

Function f	Laplace transform $\hat{f}$	Domain of $\hat{f}$
$\mathbf{1}$	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\mathbf{1}_{[0,1)}(t) + k\,\mathbf{1}_{(t=1)}$	$\frac{1-e^{-s}}{s}$	$s > 0$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \|a\|$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \|a\|$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

Table of Laplace transforms (2)

Function f	Laplace transform $\hat{f}$	Domain of $\hat{f}$
$t^n e^{at},\ n \in \mathbb{N}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t) f(t-c)$	$e^{-cs} \hat{f}(s)$	
$e^{ct} f(t)$	$\hat{f}(s-c)$	
$f(ct),\ c > 0$	$\frac{1}{c} \hat{f}(\frac{s}{c})$	
$\int_0^t f(t-\tau) g(\tau)$	$\hat{f}(s) \hat{g}(s)$	
$\delta(t-c)$	e^{-cs}	
$f^{(n)}(t)$	$s^n \hat{f}(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	
$(-t)^n f(s)$	$\hat{f}^{(n)}(s)$	

Linearity of Laplace transform

Example of function f:

$$f(t) = 5\, e^{-2t} - 3\, \sin(4t).$$

Laplace transform by linearity: we find

$$\begin{aligned} \mathcal{L}f(s) &= 5\left[\mathcal{L}(e^{-2t})\right](s) - 3\left[\mathcal{L}(\sin(4t)\right](s) \\ &= \frac{5}{s+2} - \frac{12}{s^2+16}. \end{aligned}$$

Function $f(t) = 5e^{-2t} - 3\sin(4t)$

Laplace transform

$$\boxed{\hat{f}(s)} = \mathcal{L}(5e^{-2t} - 3\sin(4t))$$

$$= 5\,\mathcal{L}(e^{-2t}) - 3\,\mathcal{L}(\sin(4t))$$

$$= 5 \times \frac{1}{s+2} - 3 \times \frac{4}{s^2+4^2}$$

$$\boxed{= \frac{5}{s+2} - \frac{12}{s^2+16}}$$

Relation between $\mathcal{L}f$ and $\mathcal{L}f'$

Theorem 10.

Hypothesis:

1. f continuous, f' piecewise continuous on $[0, A]$
   $\hookrightarrow$ for each $A > 0$.
2. $|f(t)| \leq Ke^{at}$ for $K, a \geq 0$.

Conclusion: $\mathcal{L}f'$ exists and

$$\mathcal{L}f'(s) = s\mathcal{L}f(s) - f(0)$$

Expression for $\mathcal{L}f'$

$$\boxed{\mathcal{L}f'(s)} = \int_0^\infty \overbrace{e^{-st}}^{g(t)} f'(t)\,dt$$

ibp

$$= f\,g\Big|_0^\infty - \int_0^\infty f\,\overbrace{g'}^{-se^{-st}}\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,\mathcal{L}f(s)$$

$$\boxed{= \mathcal{L}f(s) - f(0)}$$

Proof of Theorem 10

Integration by parts:

$$\int_0^A e^{-st} f'(t)\, dt = \left[e^{-st} f(t) \right]_0^A + s \int_0^A e^{-st} f(t)\, dt$$

Laplace transform of transitions

Proposition 11.

Let

- Intensities $\{\lambda_j; j \geq -1\}$, with $\lambda_{-1} = 0$
- Set of indices $\{0 \leq i, j < \infty\}$
- p_{ij} solution to forward system
  $p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t)$

Then for $i \leq j$ the Laplace transform $\hat{p}_{ij}$ satisfies

$$\hat{p}_{ij}(s) = \frac{1}{\lambda_j} \prod_{\ell=i}^{j} \frac{\lambda_\ell}{s + \lambda_\ell}$$

Proof - case $i=j$ we have seen

$$p_{ii}(t) = e^{-\lambda_i t}$$

$$\Rightarrow \quad \hat{p}_{ii}(s) = \frac{1}{s+\lambda_i}$$

Case $j>i$ then

$$p'_{ij} = \lambda_{j-1}\, p_{i,j-1} - \lambda_j\, p_{i,j}$$

$$\Rightarrow \quad \widehat{p'_{ij}} = \lambda_{j-1}\, \hat{p}_{i,j-1} - \lambda_j\, \hat{p}_{ij}$$

$$\Rightarrow \quad s\,\hat{p}_{ij} - \underbrace{p_{ij}(0)}_{=0} = \lambda_{j-1}\hat{p}_{i,j-1} - \lambda_j\, \hat{p}_{ij}$$

$$\Rightarrow \quad (s+\lambda_j)\,\hat{p}_{ij} = \lambda_{j-1}\, \hat{p}_{i,j-1}$$

$$\Rightarrow \quad \boxed{\hat{p}_{ij} = \frac{\lambda_{j-1}}{s+\lambda_j}\, \hat{p}_{i,j-1}} \longrightarrow \text{algebraic recursion}$$