Laplace transform of transitions

Proposition 11.

Let

- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$
- Set of indices $\{0 \leq i, j < \infty\}$
- p_{ij} solution to forward system
  $p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t)$

Then for $i \leq j$ the Laplace transform $\hat{p}_{ij}$ satisfies

$$\hat{p}_{ij}(s) = \frac{1}{\lambda_j} \prod_{\ell=i}^{j} \frac{\lambda_\ell}{s + \lambda_\ell}$$

Equations $\quad p_{ij}' = \lambda_{j-1} p_{i,j-1} - \lambda_j p_{ij}$

Laplace transform

$$s\hat{p}_{ij} - p_{ij}(0) = \lambda_{j-1}\hat{p}_{i,j-1} - \lambda_j \hat{p}_{ij}$$

($p_{ij}(0) = \delta_{ij}$)

$$\Leftrightarrow \quad (s+\lambda_j)\,\hat{p}_{ij} = \delta_{ij} + \lambda_{j-1}\hat{p}_{i,j-1}$$

Case $j > i$ $\quad$ we get $\delta_{ij} = 0$, and thus

$$\hat{p}_{ij} = \frac{\lambda_{j-1}}{s+\lambda_j}\hat{p}_{i,j-1}$$

$$= \frac{\lambda_{j-1}}{s+\lambda_j}\frac{\lambda_{j-2}}{s+\lambda_{j-1}}\hat{p}_{i,j-2}$$

$$= \frac{1}{\lambda_j}\frac{\lambda_j}{s+\lambda_j}\frac{\lambda_{j-1}}{s+\lambda_{j-1}}\lambda_{j-2}\hat{p}_{i,j-2}\ \ldots$$

Proof of Proposition 13 (1)

Laplace transform of the forward equation: The equation

$$p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t)$$

becomes

$$s\, \hat{p}_{ij}(s) - \delta_{ij} = \lambda_{j-1} \hat{p}_{i,j-1}(s) - \lambda_j \hat{p}_{ij}(s)$$

Rearranging terms: We get

$$(s + \lambda_j)\, \hat{p}_{ij}(s) = \delta_{ij} + \lambda_{j-1} \hat{p}_{i,j-1}(s)$$

Proof of Proposition 13 (2)

Case $j > i$: Since $\delta_{ij} = 0$ in that case, we get

$$\begin{aligned}\hat{p}_{ij}(s) &= \frac{\lambda_{j-1}}{s+\lambda_j}\,\hat{p}_{i,j-1}(s)\\ &= \frac{\lambda_{j-1}}{s+\lambda_j}\,\frac{\lambda_{j-2}}{s+\lambda_{j-1}}\,\hat{p}_{i,j-2}(s)\\ &= \frac{1}{\lambda_j}\frac{\lambda_j}{s+\lambda_j}\,\frac{\lambda_{j-1}}{s+\lambda_{j-1}}\,\lambda_{j-2}\,\hat{p}_{i,j-2}(s)\end{aligned}$$

Conclusion: Iterating the above computation, we get

$$\hat{p}_{ij}(s) = \frac{1}{\lambda_j}\prod_{\ell=i}^{j}\frac{\lambda_\ell}{s+\lambda_\ell}$$

Backward and forward system

Proposition 12.

Consider the backward system

$$\pi'_{i,j}(t) = \lambda_i \pi_{i+1,j}(t) - \lambda_i \pi_{i,j}(t)\,, \qquad (1)$$

Then

The solution $\{p_{ij};\ i,j \geq 0\}$ to the forward system also solves the system (1)

"Proof" of Prop 12   Start from equations

$$\pi'_{ij} = \lambda_i \pi_{i+1,j} - \lambda_i \pi_{ij} \qquad (2)$$

Laplace transform

$$s \hat{\pi}_{ij}(s) - \delta_{ij} = \lambda_i \hat{\pi}_{i+1,j}(s) - \lambda_i \hat{\pi}_{ij}(s) \qquad (1)$$

Then we can check that

$$\hat{p}_{ij}(s) = \frac{1}{\lambda_j} \prod_{l=i}^{j} \frac{\lambda_l}{s+\lambda_l}$$

satisfies relation (1)

$\Rightarrow$ $(p_{ij})_{i \leq j}$ satisfies backward equations

Rmk $(p_{ij})_{i \leq j}$ is one solution to (2). There might be an ∞ number of solutions

Proof of Proposition 15

Backward equation in Laplace mode: We get

$$(s+\lambda_i)\,\hat{\pi}_{ij}(s) = \delta_{ij} + \lambda_i \hat{\pi}_{i+1,j}(s) \qquad (2)$$

Forward solves backward: Take

$$\hat{\pi}_{ij}(s) = \hat{p}_{ij}(s) = \frac{1}{\lambda_j} \prod_{\ell=i}^{j} \frac{\lambda_\ell}{s+\lambda_\ell}$$

This solves (2)

Problem with the backward system

Main problem:

Backward system may not have a unique solution

Minimal solution:

The unique solution of the forward system
is a minimal solution of the backward system

Minimal solution of the backward system

Proposition 13.

Let

- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$
- Set of indices $\{0 \leq i, j < \infty\}$
- p_{ij} solution to forward system
  $p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t)$

Then

$$\pi_{ij} \text{ solution of the backward system}$$
$$\Longrightarrow \text{We have } p_{i,j}(t) \leq \pi_{i,j}(t) \text{ for all } i, j \in S \text{ and } t \geq 0$$

Backward system and explosion

Then (p_{ij}) is also the largest possible solution to the backward system

Relating explosion time and uniqueness: ⇒ the smallest solution is also the largest

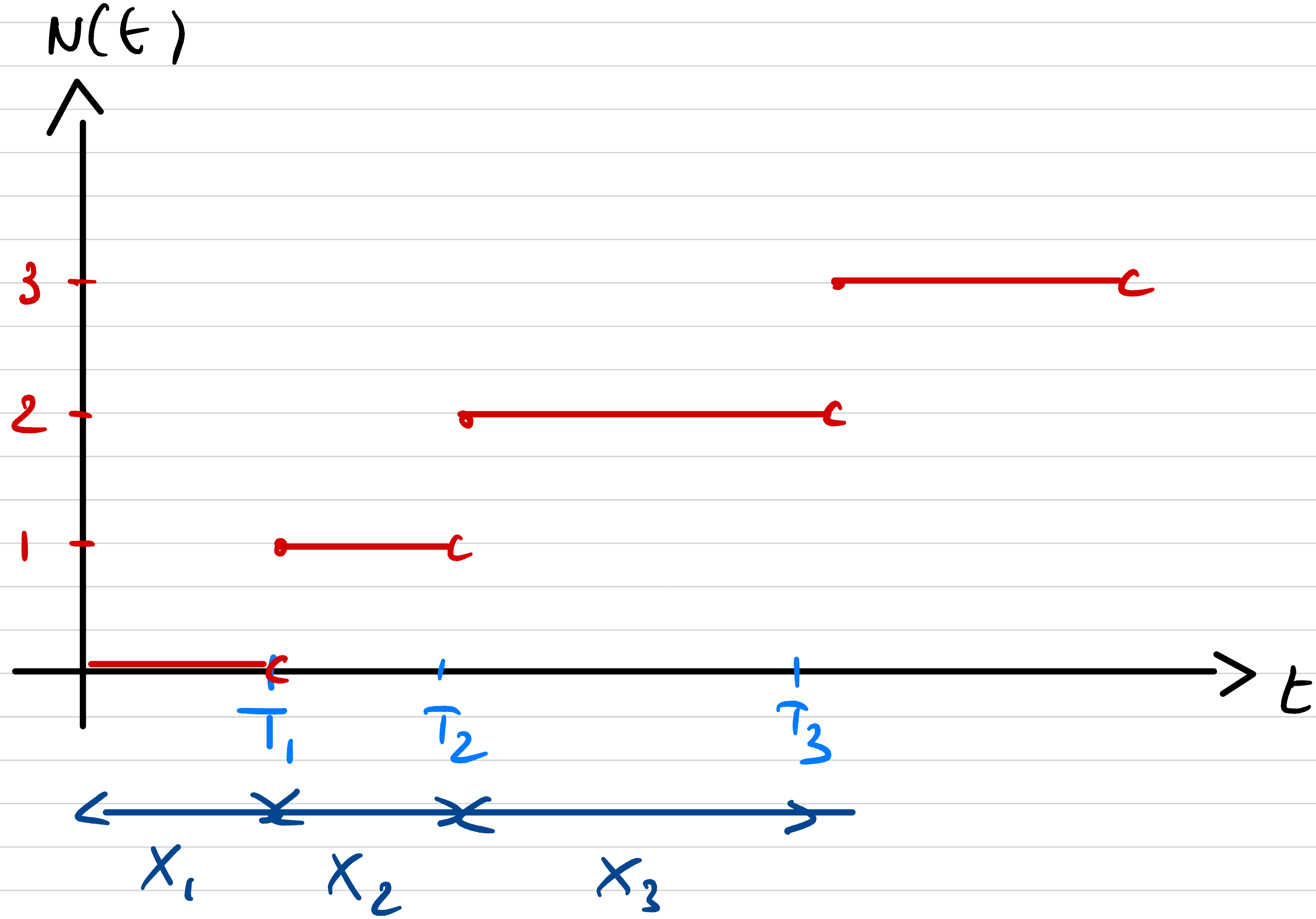
1. If $\sum_{j \in S} p_{i,j}(t) = 1$, then
   $\hookrightarrow$ $p_{i,j}$ is the unique solution of the backward system ⇒ ∃! solution to backward
2. Problem: $\{p_{i,j}(t);\, j \in S\}$ is not always a distribution
3. This is related to explosion time: we might have

$$\mathbf{P}\left(T_\infty < \infty\right) > 0, \quad \text{where} \quad T_\infty = \lim_{n \to \infty} T_n$$

T_n = n-th arrival time

Then solution to backward system is non unique

Next question: Do we have $P(T_\infty < \infty) = 0$?



The X_i's are $\perp\!\!\!\perp$, and $X_i \sim \mathcal{E}(\lambda_{i-1})$

Honest birth process

Definition 14.

Let

- N birth process
- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$
- $\{T_n;\, n \geq 1\}$ arrival times

Then N is said to be honest if

$$\mathbf{P}\left(T_\infty = \infty\right) = 1$$

(no explosion in finite time)

Sum of exponential random variables

Proposition 15.

Let

Application: x_n = inter-arrival time

- $\{X_n; n \geq 1\}$ sequence of independent random variables
- Each X_n is such that $X_n \sim \mathcal{E}(\lambda_{n-1})$
- $T_\infty = \sum_{n=1}^{\infty} X_n$

Then

$$\mathbf{P}(T_\infty < \infty) = \begin{cases} 0, & \text{if } \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty \end{cases}$$

Setting $(X_n)_{n\geq 1}$, $\perp\!\!\!\perp$, $X_n \sim \mathcal{E}(\lambda_{n-1})$

Easy case: $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty$

In that case one can prove that

$$\boxed{\mathbb{E}\left[\sum_{n=1}^{\infty} X_n\right] < \infty} \Rightarrow \mathbb{P}\left(\sum_{n=1}^{\infty} X_n < \infty\right) = 1$$

Here

$$\mathbb{E}\left[\sum_{n=1}^{\infty} X_n\right] \underset{\text{Fubini-Tonelli}}{\overset{X_n \geq 0}{=}} \sum_{n=1}^{\infty} \mathbb{E}[X_n]$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lambda_{n-1}} < \infty \text{ by assumption}$$

Conclusion If $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty$, then

$$\mathbb{P}\left(\sum_{n=1}^{\infty} X_n < \infty\right) = 1 \quad (\Rightarrow \text{explosion in finite time})$$

Proof of Proposition 15 (1)

Case $\sum_{n\geq 1} \lambda_n^{-1} < \infty$: Using Fubini-Tonelli we have

$$\mathbf{E}\left[T_\infty\right] = \mathbf{E}\left[\sum_{n=1}^{\infty} X_n\right] = \sum_{n=1}^{\infty} \frac{1}{\lambda_{n-1}} < \infty$$

Thus

$$\mathbf{P}\left(T_\infty < \infty\right) = 0$$

Proof of Proposition 15 (2)

Case $\sum_{n\geq 1} \lambda_n^{-1} = \infty$, strategy: We have

$$\mathbf{E}\left[e^{-T_\infty}\right] = 0 \implies \mathbf{P}\left(e^{-T_\infty} = 0\right) = 1$$
$$\implies \mathbf{P}\left(T_\infty = \infty\right) = 1$$

We will thus prove

$$\mathbf{E}\left[e^{-T_\infty}\right] = 0$$

Proof of Proposition 15 (3)

Case $\sum_{n\geq 1} \lambda_n^{-1} = \infty$, computation: We have

$$
\begin{aligned}
\mathbf{E}\left[e^{-T_\infty}\right] &= \mathbf{E}\left[\prod_{n=1}^{\infty} e^{-X_n}\right] \\
&= \lim_{N\to\infty} \mathbf{E}\left[\prod_{n=1}^{N} e^{-X_n}\right] \qquad \text{(monotone convergence)} \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} \mathbf{E}\left[e^{-X_n}\right] \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} \frac{1}{1+\lambda_{n-1}^{-1}} = \left(\prod_{n=1}^{\infty}\left(1+\frac{1}{\lambda_{n-1}}\right)\right)^{-1}
\end{aligned}
$$

Proof of Proposition 15 (4)

Infinite products: If $u_n \geq 0$, then

$$\prod_{n=1}^{\infty} (1+u_n) = \infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} u_n = \infty \tag{3}$$

Pseudo-proof of (3): We have

$$\begin{aligned} \ln\left(\prod_{n=1}^{\infty}(1+u_n)\right) &= \sum_{n=1}^{\infty} \ln(1+u_n) \\ &\asymp \sum_{n=1}^{\infty} u_n \end{aligned}$$

Proof of Proposition 15 (5)

Recall: We have seen

$$\mathbf{E}\left[e^{-T_\infty}\right] = \left(\prod_{n=1}^{\infty}\left(1+\frac{1}{\lambda_{n-1}}\right)\right)^{-1}$$

Application of (3):

$$\mathbf{E}\left[e^{-T_\infty}\right] \iff \prod_{n=1}^{\infty}\left(1+\frac{1}{\lambda_{n-1}}\right)=\infty \iff \sum_{n\geq 1}\lambda_n^{-1}=\infty$$

Conclusion:

$$T_\infty=\infty \quad\iff\quad \sum_{n\geq 1}\lambda_n^{-1}=\infty$$