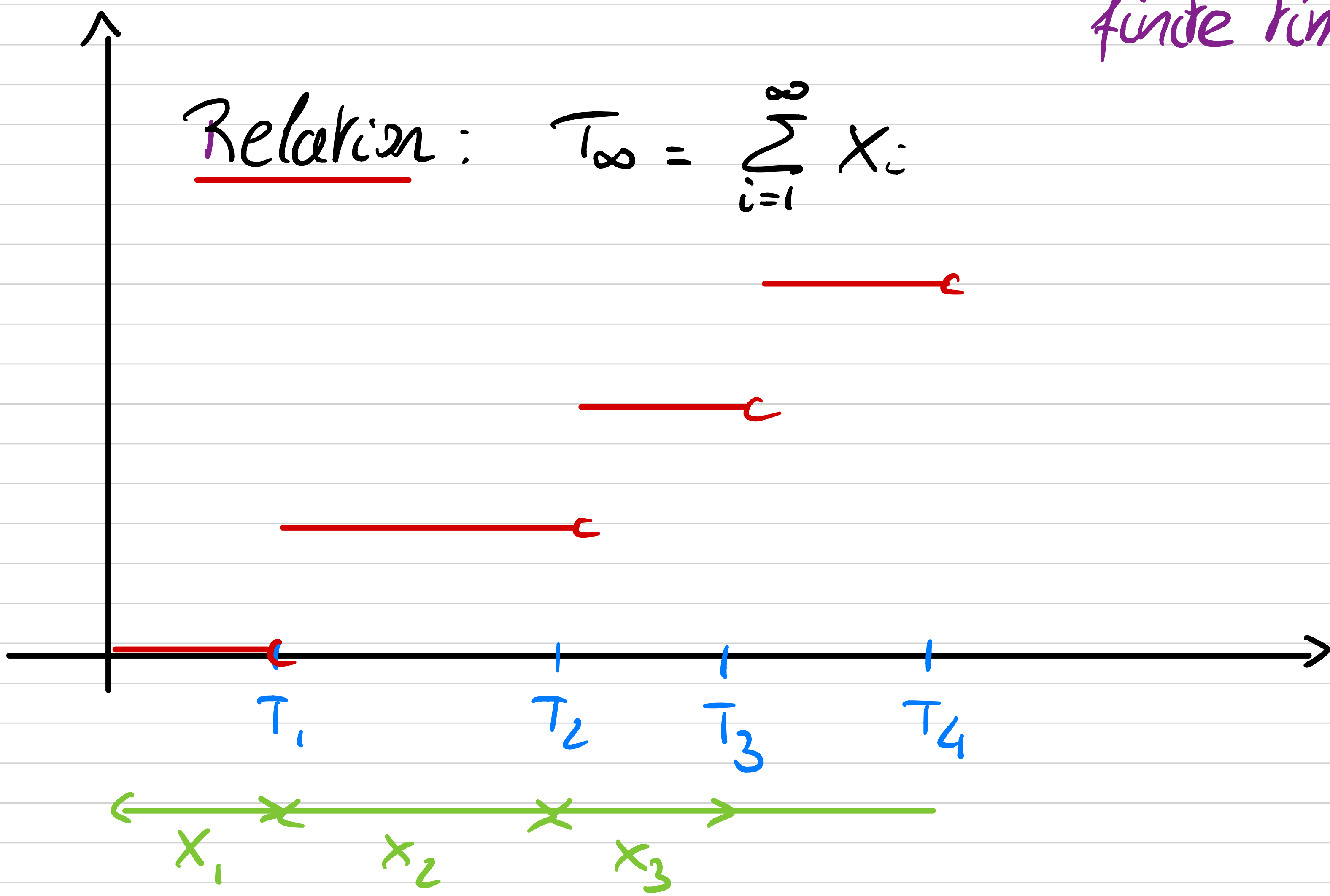
Question : Is $T_\infty < \infty$? (Explosion in finite time)

Relation : $T_\infty = \sum_{i=1}^{\infty} X_i$



We know that X_i's are $\perp\!\!\!\perp$, $X_i \sim \mathcal{E}(\lambda_{i-1})$

Sum of exponential random variables

Proposition 15.

Let

- $\{X_n; n \geq 1\}$ sequence of independent random variables
- Each X_n is such that $X_n \sim \mathcal{E}(\lambda_{n-1})$
- $T_\infty = \sum_{n=1}^{\infty} X_n$

Then

$$\mathbf{P}\left(T_\infty < \infty\right) = \begin{cases} 0, & \text{if } \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty \end{cases}$$

Proof of Proposition 15 (1)

Case $\sum_{n\geq 1} \lambda_n^{-1} < \infty$: Using Fubini-Tonelli we have

$$\mathbf{E}\left[T_\infty\right] = \mathbf{E}\left[\sum_{n=1}^{\infty} X_n\right] = \sum_{n=1}^{\infty} \frac{1}{\lambda_{n-1}} < \infty$$

Thus

$$\mathbf{P}\left(T_\infty < \infty\right) = 0$$

Case $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$. In that case, we consider a kind of Laplace transform:

$$E[e^{-T_\infty}]$$

e^{-T_∞} is a ≥ 0 r.v. If $E[e^{-T_\infty}]=0$, then

$$\boxed{P(e^{-T_\infty} = 0) = 1}$$

or $E[e^{-T_\infty}] = 0 \Rightarrow e^{-T_\infty} = 0$ a.s

$$P(T_\infty = \infty) = 1$$

Reduction: It is enough to prove

$$\boxed{E[e^{-T_\infty}] = 0}$$

Computation for $E[e^{-T_\infty}]$

$$E[e^{-T_\infty}] = E\left[\exp\left(-\sum_{n=1}^{\infty} X_n\right)\right]$$

$$= E\left[\prod_{n=1}^{\infty} e^{-X_n}\right] \overset{?}{=} \prod_{n=1}^{\infty} E[e^{-X_n}]$$

$$= E\left[\lim_{N\to\infty} \prod_{n=1}^{N} e^{-X_n}\right]$$

dominated / monotone convergence

$$= \lim_{N\to\infty} E\left[\prod_{n=1}^{N} e^{-X_n}\right]$$

$$= \lim_{N\to\infty} \prod_{n=1}^{N} E[e^{-X_n}]$$

Next

$$E[e^{-X_n}] = \int_0^\infty \overbrace{\lambda_{n-1} e^{-\lambda_{n-1} t}}^{\text{density } \mathcal{E}(\lambda_{n-1})} e^{-t}\, dt$$

$$= \lambda_{n-1} \int_0^\infty e^{-(\lambda_{n-1}+1)t}\, dt = \frac{\lambda_{n-1}}{\lambda_{n-1}+1} = \frac{1}{1+1/\lambda_{n-1}}$$

Summary

$$E[e^{-T_\infty}] = \lim_{N\to\infty} \prod_{n=1}^{N} \frac{1}{1+1/\alpha_{n-1}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1+1/\alpha_{n-1}}$$

$$= \frac{1}{\prod_{n=1}^{\infty}\left(1+\frac{1}{\alpha_{n-1}}\right)}$$

2nd reduction

$= u_n$

$$E[e^{-T_\infty}] = 0 \iff \prod_{n=1}^{\infty}\left(1+\frac{1}{\alpha_{n-1}}\right) = \infty$$

Claim If $u_n \geq 0$, then

$$\prod_{n=1}^{\infty}(1+u_n) = \infty \iff \sum u_n = \infty$$

"Proof" of last claim

hyp: u_n small

$$\ln\left(\prod_{n=1}^{\infty}(1+u_n)\right) = \sum_{n=1}^{\infty} \ln(1+u_n)$$

$$\simeq \sum_{n=1}^{\infty} u_n$$

$\left(\frac{1}{2}u \leq \ln(1+u) \leq u \quad \text{if } u \text{ small} \right)$

Proof of Proposition 15 (2)

Case $\sum_{n\geq 1}\lambda_n^{-1}=\infty$, strategy: We have

$$\begin{aligned}\mathbf{E}\left[e^{-T_\infty}\right]=0 \quad&\Longrightarrow\quad \mathbf{P}\left(e^{-T_\infty}=0\right)=1\\ &\Longrightarrow\quad \mathbf{P}\left(T_\infty=\infty\right)=1\end{aligned}$$

We will thus prove

$$\mathbf{E}\left[e^{-T_\infty}\right]=0$$

Proof of Proposition 15 (3)

Case $\sum_{n\geq 1} \lambda_n^{-1} = \infty$, computation: We have

$$
\begin{aligned}
\mathbf{E}\left[e^{-T_\infty}\right] &= \mathbf{E}\left[\prod_{n=1}^{\infty} e^{-X_n}\right] \\
&= \lim_{N\to\infty} \mathbf{E}\left[\prod_{n=1}^{N} e^{-X_n}\right] \qquad \text{(monotone convergence)} \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} \mathbf{E}\left[e^{-X_n}\right] \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} \frac{1}{1+\lambda_{n-1}^{-1}} = \left(\prod_{n=1}^{\infty}\left(1+\frac{1}{\lambda_{n-1}}\right)\right)^{-1}
\end{aligned}
$$

Proof of Proposition 15 (4)

Infinite products: If $u_n \geq 0$, then

$$\prod_{n=1}^{\infty}(1+u_n)=\infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty}u_n=\infty \tag{3}$$

Pseudo-proof of (3): We have

$$\begin{aligned} \ln\left(\prod_{n=1}^{\infty}(1+u_n)\right) &= \sum_{n=1}^{\infty}\ln(1+u_n) \\ &\asymp \sum_{n=1}^{\infty}u_n \end{aligned}$$

Proof of Proposition 15 (5)

Recall: We have seen

$$\mathbf{E}\left[e^{-T_\infty}\right] = \left(\prod_{n=1}^{\infty}\left(1+\frac{1}{\lambda_{n-1}}\right)\right)^{-1}$$

Application of (3):

$$\mathbf{E}\left[e^{-T_\infty}\right] \iff \prod_{n=1}^{\infty}\left(1+\frac{1}{\lambda_{n-1}}\right) = \infty \iff \sum_{n\geq 1}\lambda_n^{-1} = \infty$$

Conclusion:

$$T_\infty = \infty \quad \iff \quad \sum_{n\geq 1}\lambda_n^{-1} = \infty$$

Application to birth process

Proposition 16.

Let

- N birth process
- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$
- $\{T_n;\, n \geq 1\}$ arrival times

Then N is honest iff

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$$

Final remarks

Notes before next section:

1. Poisson and birth processes are Markov processes
   $\hookrightarrow$ Due to $(N(t) - N(s)) \perp\!\!\!\perp$ Past, given $N(s) = i$

2. They are in fact strong Markov processes
   $\hookrightarrow$ Definition to be seen later

3. Problems can occur due to explosions
   $\hookrightarrow$ This could not be observed in discrete time

Outline

Outline

Vocabulary

Stochastic process:

- Family $\{X(t);\, t \in [0, \infty)\}$ of random variables
- Family evolving in a random but prescribed manner
- Here $X(t) \in S$, where S countable state space with $N = |S|$

Markov evolution:

Conditioned on $X(t)$,
the evolution does not depend on the past

Markov chain

Definition 17.

Let

- $X = \{X(t);\ t \geq 0\}$ stochastic process

We say that X is a continuous time Markov chain if

set of >0 probab

$$\mathbf{P}\left(X(t_n) = j \,|\, X(t_1) = i_1, \ldots, X(t_{n-1}) = i_{n-1}\right)$$
$$= \mathbf{P}\left(X(t_n) = j \,|\, X(t_{n-1}) = i_{n-1}\right),$$

discrete r.v.

for all

- $0 \leq t_1 < \cdots < t_n < \infty$
- $i_1, \ldots, i_n, j \in S$

Differences with discrete time

Main difference:

- No time unit
- Therefore no exact analogue of P

Method 1:

- Use infinitesimal calculus
- This leads to infinitesimal generator

Method 2:

- Embedded chain $\{X_{T_n};\ T_n \text{ arrival times}\}$ is usually a discrete Markov chain

Birth process as Markov process

Proposition 18.

Let

- N birth process
- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$

Then

$$N \text{ is a Markov process}$$