Birth process as Markov process

Proposition 18.

Let

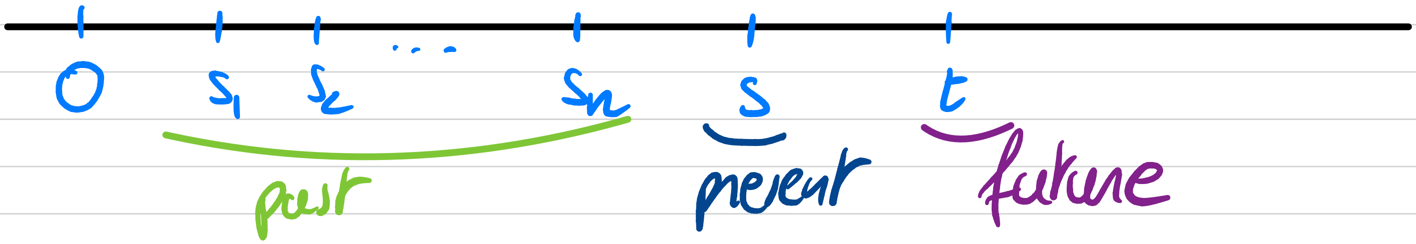
- N birth process
- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$

Then

N is a Markov process

Hyp: $N(t)-N(s) \perp\!\!\!\perp$ "past" conditioned on $N(s)$

Proof for birth process



we wish

$$\mathbb{P}(N(t)=j \mid N(s_1)=i_1,\dots,N(s_n)=i_n, N(s)=i)$$

$$= \mathbb{P}(N(t)=j \mid N(s)=i)$$

Equivalent wish

$$\mathbb{P}(N(t)-N(s)=j-i \mid N(s_1)=i_1,\dots,N(s_n)=i_n, N(s)=i)$$

$$= \mathbb{P}(N(t)-N(s)=j-i \mid N(s)=i)$$

Hyp: $N(t)-N(s) \perp\!\!\!\perp$ "past" conditioned on $N(s)$

<u>wish</u>

$$\mathbb{P}\big(\overbrace{N(t)-N(s) = j-i}^{A_{st}} \mid \overbrace{N(s_1)=i_1, \dots, N(s_n)=i_n}^{B_{s_1\dots s_n}}, \overbrace{N(s)=i}^{C_s}\big)$$

$$= \mathbb{P}\left(N(t)-N(s) = j-i \mid N(s)=i\right)$$

<u>we want to prove</u>

$$\mathbb{P}(A_{st} \mid B_{s_1\dots s_n} \cap C_s) = \mathbb{P}(A_{st} \mid C_s)$$

<u>we know</u>

(i) $\mathbb{P}(A_{st} \cap B_{s_1\dots s_n} \mid C_s)$ (general formula)

$$= \mathbb{P}(A_{st} \mid B_{s_1\dots s_n} \cap C_s) \quad \mathbb{P}(B_{s_1\dots s_n} \mid C_s)$$

definition of cond. $\perp\!\!\!\perp$

(ii) $\mathbb{P}(A_{st} \cap B_{s_1\dots s_n} \mid C_s) = \mathbb{P}(A_{st} \mid C_s)\, \mathbb{P}(B_{s_1\dots s_n} \mid C_s)$

Conclusion we have seen

(i) $\mathbb{P}(A_{st} \cap B_{s_1 \dots s_n} \mid C_s)$ (general formula)

$= \mathbb{P}(A_{st} \mid B_{s_1 \dots s_n} \cap C_s) \quad \mathbb{P}(B_{s_1 \dots s_n} \mid C_s)$

definition of cond. $\perp\!\!\!\perp$

(ii) $\mathbb{P}(A_{st} \cap B_{s_1 \dots s_n} \mid C_s) = \mathbb{P}(A_{st} \mid C_s) \, \mathbb{P}(B_{s_1 \dots s_n} \mid C_s)$

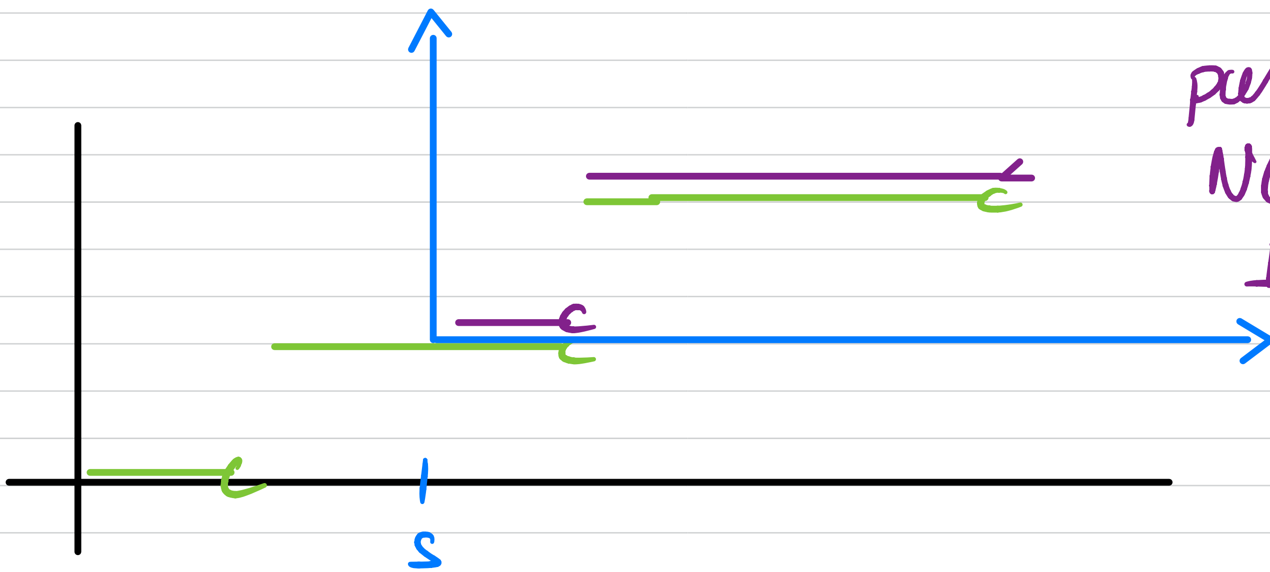
$\Rightarrow \quad \mathbb{P}(A_{st} \mid B_{s_1 \dots s_n} \cap C_s) \, \mathbb{P}(B_{s_1 \dots s_n} \mid C_s)$

$= \mathbb{P}(A_{st} \mid C_s) \quad \mathbb{P}(B_{s_1 \dots s_n} \mid C_s)$

$$\Rightarrow \quad \boxed{\mathbb{P}(A_{st} \mid B_{s_1 \dots s_n} \cap C_s) = \mathbb{P}(A_{st} \mid C_s)}$$

Rmk If N is Poisson process, intensity λ.
Then

$$N(t)-N(s) \perp\!\!\!\perp \text{past}$$



purple process
$N(t)-N(s)$
$\perp\!\!\!\perp$ past

For a birth process, we need to condition on $N(s)$. Why? $\longrightarrow$ we need to know $N(s)$ to know the intensities for $N(t)-N(s)$

Independence $\quad A \perp\!\!\!\perp B$ if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

Cond. $\perp\!\!\!\perp$: $\quad A \perp\!\!\!\perp B$ condit. on C if

$$\boxed{\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\,\mathbb{P}(B \mid C)}$$

Proof of Proposition 18 (1)

Setting: Consider

- $s_1 < \cdots < s_n < s < t$
- $i_1, \ldots, i_n, j \in S$

Aim: Prove

$$\begin{aligned} &\mathbf{P}\left(N(t) = j \middle| N(s_1) = i_1, \ldots, N(s_n) = i_n, N(s) = i\right) \\ =&\mathbf{P}\left(N(t) = j \middle| N(s) = i\right) \end{aligned}$$

Equivalent statement: Prove that

$$\begin{aligned} &\mathbf{P}\left(N(t) - N(s) = j - i \middle| N(s_1) = i_1, \ldots, N(s_n) = i_n, N(s) = i\right) \\ =&\mathbf{P}\left(N(t) - N(s) = j - i \middle| N(s) = i\right) \end{aligned}$$

Proof of Proposition 18 (2)

Recall: We wish to prove

$$\begin{aligned} &\mathbf{P}\left(N(t)-N(s)=j-i \,\middle|\, N(s_1)=i_1,\ldots,N(s_n)=i_n, N(s)=i\right) \\ =&\mathbf{P}\left(N(t)-N(s)=j-i \,\middle|\, N(s)=i\right) \end{aligned}$$

Defining some sets: Consider

- $A_{st}=(N(t)-N(s)=j-i)$
- $B_{s_1,\ldots,s_n}=N(s_1)=i_1,\ldots,N(s_n)=i_n$
- $C_s=(N(s)=i)$

Rephrasing our claim: Now we wish to prove

$$\mathbf{P}\left(A_{st} \,\middle|\, B_{s_1,\ldots,s_n}\cap C_s\right)=\mathbf{P}\left(A_{st} \,\middle|\, C_s\right)$$

Proof of Proposition 18 (3)

General formula: We have

$$\mathbf{P}\left(A_{st} \cap B_{s_1,\ldots,s_n} \middle| C_s\right) = \mathbf{P}\left(A_{st} \middle| B_{s_1,\ldots,s_n} \cap C_s\right) \mathbf{P}\left(B_{s_1,\ldots,s_n} \middle| C_s\right) \tag{4}$$

Conditional independence: In Definition 5 we had the assumption

$$\text{Conditional on } N(s),\ N(t) - N(s) \perp\!\!\!\perp \text{ values of } N \text{ on } [0, s]$$

This reads

$$\mathbf{P}\left(A_{st} \cap B_{s_1,\ldots,s_n} \middle| C_s\right) = \mathbf{P}\left(A_{st} \middle| C_s\right) \mathbf{P}\left(B_{s_1,\ldots,s_n} \middle| C_s\right) \tag{5}$$

Conclusion: Combining (4) and (5) we end up with

$$\mathbf{P}\left(A_{st} \middle| B_{s_1,\ldots,s_n} \cap C_s\right) = \mathbf{P}\left(A_{st} \middle| C_s\right)$$

Transition probabilities

Definition 19.

Let X be a continuous-time Markov chain. Then

1. The transition probabilities are given by

$$p_{ij}(s,t) = \mathbf{P}\left(X(t) = j \,|\, X(s) = i\right) \quad \text{for} \quad s < t, \; i,j \in S$$

2. X is homogeneous if for all $\overset{s,t}{\cancel{x}}, i, j$ we have

$$p_{ij}(s,t) = p_{ij}(0, t-s) \equiv p_{ij}(t-s)$$

Hypothesis 20.

In the chapter we always assume that X is homogeneous

Transitions for the Poisson process

Proposition 21.

Let

- N Poisson process
- Intensity λ

Then N is homogeneous and

$$p_{ij}(s,t) = p_{ij}(t-s) = \exp(-\lambda(t-s))\,\frac{(\lambda(t-s))^{j-i}}{(j-i)!}$$

Transition for Poisson

$$\boxed{p_{ij}(s,t)} = \mathbb{P}(N(t) = j \mid N(s) = i)$$

$$= \mathbb{P}(N(t) - N(s) = j-i \mid N(s) = i)$$

$$= \mathbb{P}(N(t) - N(s) = j-i) \qquad (N(t) - N(s) \perp\!\!\!\perp N(s))$$

We know : $N(t) - N(s) = \hat{N}(t-s)$ is also a PP with intensity λ. We have

$$\hat{N}(t-s) \overset{(d)}{=} \mathcal{P}(\lambda(t-s))$$

$$\boxed{= e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{j-i}}{(j-i)!}}$$