Outline

Continuity of standard semigroups

Proposition 25.

Assume

- X Markov chain
- The transition P is standard

Then P is continuous: for all $t \geq 0$ we have

$$\lim_{h \to 0} P_{t+h} = P_t,$$

that is

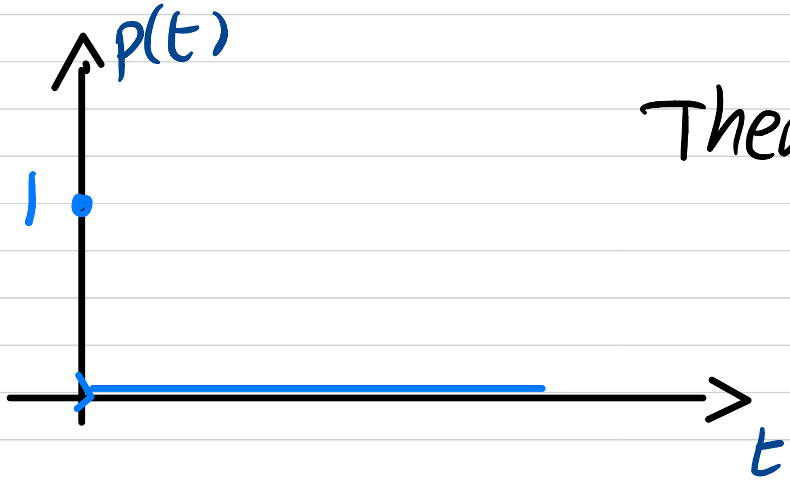
$$\lim_{h \to 0} p_{ij}(t+h) = p_{ij}(t), \text{ for all } i, j \in S$$

Rmk Example of non standard P_t

we take $S=\{1\}$ and $p:[0,\infty)\to[0,1]$

$p(0) = 1$

$p(t) = 0$ if $t>0$



Then

$$P_t P_s = P_{t+s}$$

In general

$$p_{ij}(t) = P(X(\overset{\text{time}}{t}) = \underset{}{j} \mid X(\overset{\text{time}}{0}) = i)$$

time

$\in S$

time

$\in S$

Behavior close to 0

Taylor expansions: We have (admitted)

$$\begin{aligned} p_{ij}(h) &= g_{ij}h + o(h) \\ p_{ii}(h) &= 1 + g_{ii}h + o(h) \end{aligned}$$

Signs of g_{ij}: If we want $p_{ij}(h) \in [0, 1]$ we need

$$g_{ij} \geq 0, \quad \text{and} \quad g_{ii} \leq 0$$

Meaning of g_{ij}'s

Interpretation: Starting from $X(t) = i$,

1. Nothing happens with probability

$$\mathbb{P}(X(t+h)=i \,|\, X(t)=i) \simeq 1 + g_{ii}h + o(h)$$

2. The chain jumps from i to j with probability

$$\mathbb{P}(X(t+h)=j \,|\, X(t)=i) \simeq g_{ij}h + o(h)$$

Terminology:

The matrix $G = (g_{ij})_{i,j \in S}$ is called generator of the Markov chain

Basic property of the generator

Proposition 26.

$$\sum_{j \in S} p_{ij}(t) = 1$$

Assume

- X Markov chain
- The transition P is standard
- There is a generator G

$$\sum_{j \in S} g_{ij} = \underbrace{g_{ii}}_{\leq 0} + \underbrace{\sum_{j \neq i} g_{ij}}_{\geq 0}$$

Then for most cases we have

$$\sum_{j \in S} g_{ij} = 0, \quad \text{for all} \quad i \in S$$

Generator for birth process

Proposition 27.

Let

- N birth process
- Intensities $\{\lambda_j; \, j \geq -1\}$, with $\lambda_{-1} = 0$

Then the generator G of N is given by

$$g_{ii} = -\lambda_i, \quad g_{i,i+1} = \lambda_i, \quad g_{ij} = 0 \text{ otherwise}, \tag{6}$$

that is

$$G = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ 0 & -\lambda_1 & \lambda_1 & 0 & 0 & \cdots \\ 0 & 0 & -\lambda_2 & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Expression of G for birth process

General expression for Markov chain

$$p_{ii}(h) = 1 + g_{ii} h + o(h)$$

$$p_{ij}(h) = g_{ij} h + o(h) \qquad j \neq i$$

For birth processes, we have seen

$$p_{ii}(h) = 1 - \lambda_i h + o(h)$$

$$p_{i,i+1}(h) = \lambda_i h + o(h)$$

$$p_{i,j}(h) = o(h) \quad \text{if } j \neq i, i+1$$

We identify :

$$g_{ii} = -\lambda_i \qquad g_{ij} = 0 \text{ if } j \neq i, i+1$$

$$g_{i,i+1} = \lambda_i$$

Proof of Proposition 27

Expansion for birth transitions: We have seen (cf Definition 5)

$$\begin{aligned} p_{n,n}(t,t+h) &= p_{n,n}(h) = 1 - \lambda_n h + o(h) \\ p_{n,n+1}(t,t+h) &= p_{n,n+1}(h) = \lambda_n h + o(h) \\ p_{n,j}(t,t+h) &= p_{n,j}(h) = o(h)\,, \quad \text{if } j \geq n+2 \end{aligned}$$

General expansion: We have also seen the general expression

$$\begin{aligned} p_{nn}(h) &= 1 + g_{nn} h + o(h) \\ p_{nj}(h) &= g_{nj} h + o(h) \end{aligned}$$

Conlusion:
We easily get (6) by identification

Matrix form of the generator

Proposition 28.

Assume

- X Markov chain
- The transition P is standard

Then we have

$G = P'_t|_{t=0}$

$$\lim_{h\to 0} \frac{1}{h}\left(P_h - \mathrm{Id}\right) = G\,,$$

that is

$$\lim_{h\to 0} \frac{1}{h}\left(p_{ij}(h) - \delta_{ij}\right) = g_{ij}, \quad \text{for all } i,j \in S$$

$p_{ij}(h) = \mathbb{P}(X(h) = j \mid X(0) = i)$

Proof

$$p_{ii}(h) = 1 + g_{ii} h + o(h)$$

$$p_{ij}(h) = g_{ij} h + o(h)$$

Thus

$$\frac{p_{ij}(h) - \mathbb{1}_{(i=j)}}{h} = \frac{g_{ij} h + o(h)}{h}$$

$$\boxed{\lim_{h \to 0} \frac{1}{h}\left(p_{ij}(h) - \mathbb{1}_{(i=j)}\right) = g_{ij}}$$

Proof of Proposition 28

Main argument: Rephrasing of

$$\begin{aligned} p_{ij}(h) &= g_{ij}h + o(h) \\ p_{ii}(h) &= 1 + g_{ii}h + o(h) \end{aligned}$$

Transitions from generator: forward equations

Proposition 29.

Assume

- X Markov chain
- The transition P is standard

Then P_t satisfies the differential equation

$$P_t' = P_t \, G \, .$$

kj

that is

$$p_{ij}'(t) = \sum_{k \in S} p_{ik}(t) g_{jk}, \quad \text{for all } i, j \in S$$

Proof Use Chapman-K

$$p_{ij}(t+h) = \sum_{k\in S} p_{ik}(t)\, p_{kj}(h)$$

$$= \sum_{k\neq j} p_{ik}(t)\,(g_{kj}h + o(h))$$

$$+ \; p_{ij}(t)\,(1+g_{jj}h + o(h))$$

Thus

$$p_{ij}(t+h) - p_{ij}(t) = \sum_{k\in S} p_{ik}(t)\, g_{kj}\, h + o(h)$$

$$\lim_{h\to 0} \frac{1}{h}\left(p_{ij}(t+h) - p_{ij}(t)\right) = \lim_{h\to 0} \sum_{k\in S} p_{ik}(t)\, g_{kj} + o(1)$$

$$\boxed{p_{ij}'(t) = \sum_{k\in S} p_{ik}(t)\, g_{kj}}$$

Proof of Proposition 29

Application of Chapman-Kolmogorov:

$$\begin{aligned} p_{ij}(t+h) &= \sum_{k\in S} p_{ik}(t)p_{kj}(h) \\ &\simeq p_{ij}(t)\,(1+g_{jj}h) + \sum_{k\neq j} p_{ik}(t)g_{kj}h \\ &= p_{ij}(t) + \sum_{k\in S} p_{ik}(t)g_{kj}h \end{aligned}$$

Differentiating:

$$\frac{1}{h}\left(p_{ij}(t+h) - p_{ij}(t)\right) \simeq \sum_{k\in S} p_{ik}(t)g_{kj} = (P_t G)_{ij}$$