We have seen

$$\begin{cases} P'_t = P_t G \\ P_0 = Id \end{cases}$$

, and $t \mapsto P_t$ is matrix-valued

If $t \mapsto y_t$ is $\mathbb{R}$-valued, and

$$\begin{cases} y'_t = y_t a = a y_t \\ y_0 = 1 \end{cases}$$

we know that $y_t = e^{at} = \sum_{n=0}^{\infty} \frac{t^n}{n!} a^n$

$\boxed{\text{This is also true for matrices}}$

Transitions from generator: matrix exponential

Proposition 30.

Assume

- X Markov chain
- The transition P is standard

Then P_t satisfies the relation

$$P_t = e^{t\,G}, \quad \text{where} \quad e^{t\,A} \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

Pseudo-proof We have $P_t' = P_t G = G P_t$

Thus

$$\boxed{P_t - P_s} = \int_s^t P_r' \, dr$$

$$= \int_s^t G P_r \, dr = \int_s^t G(P_r - P_s + P_s) \, dr$$

$$= G(t-s) P_s + \int_s^t G(P_{r_1} - P_s) \, dr_1$$

$$= G(t-s) P_s + \int_s^t G\left(\int_s^{r} P_{r_2}' \, dr_2\right) dr_1$$

$$= G(t-s) P_s + \int_s^t G \int_s^{r_1} G(P_{r_2} - P_s + P_s) \, dr_2 \, dr_1$$

$$= G(t-s) P_s + G^2 P_s \int_s^t \int_s^{r_1} dr_2 \, dr_1 + \text{remainder}$$

$$\boxed{= G(t-s) P_s + G^2 P_s \frac{t^2}{2} + \text{remainder}}$$

Iterating, we get $\sum \frac{(t-s)^n}{n!} G^n$

General inter-arrival

Proposition 31.

Let

- X Markov chain with transition P_t
- U random variable defined by

$$U = \inf\{t \geq 0; X(s+t) \neq i\}$$

Then we have

$$(-g_{ii})\, e^{g_{ii} r}\, dr$$

$$\mathcal{L}(U|\, X(s) = i) = \overset{\frown}{\mathcal{E}(-g_{ii})},$$

that is

$$= \int_t^\infty (-g_{ii})\, e^{g_{ii} r}\, dr$$

$$\mathbf{P}(U > t|\, X(s) = i) = \exp(\cancel{-}g_{ii}\, t)$$

$U \in \mathbb{R}$

Conditional law

If $A \subset \Omega$ is an event

U random variable

law or distribution

we say that $\mathcal{L}(U \mid A) = f_A(x)\,dx$
if

$$\boxed{\mathbb{E}[\,g(U) \mid A\,] = \int g(x)\, f_A(x)\, dx}$$

$\forall g$ bounded

In fact it is enough to check

$$\boxed{\mathbb{P}(\,U > t \mid A\,) = \int_t^{\infty} f_A(x)\, dx} \quad \forall t$$

Gaussian distribution

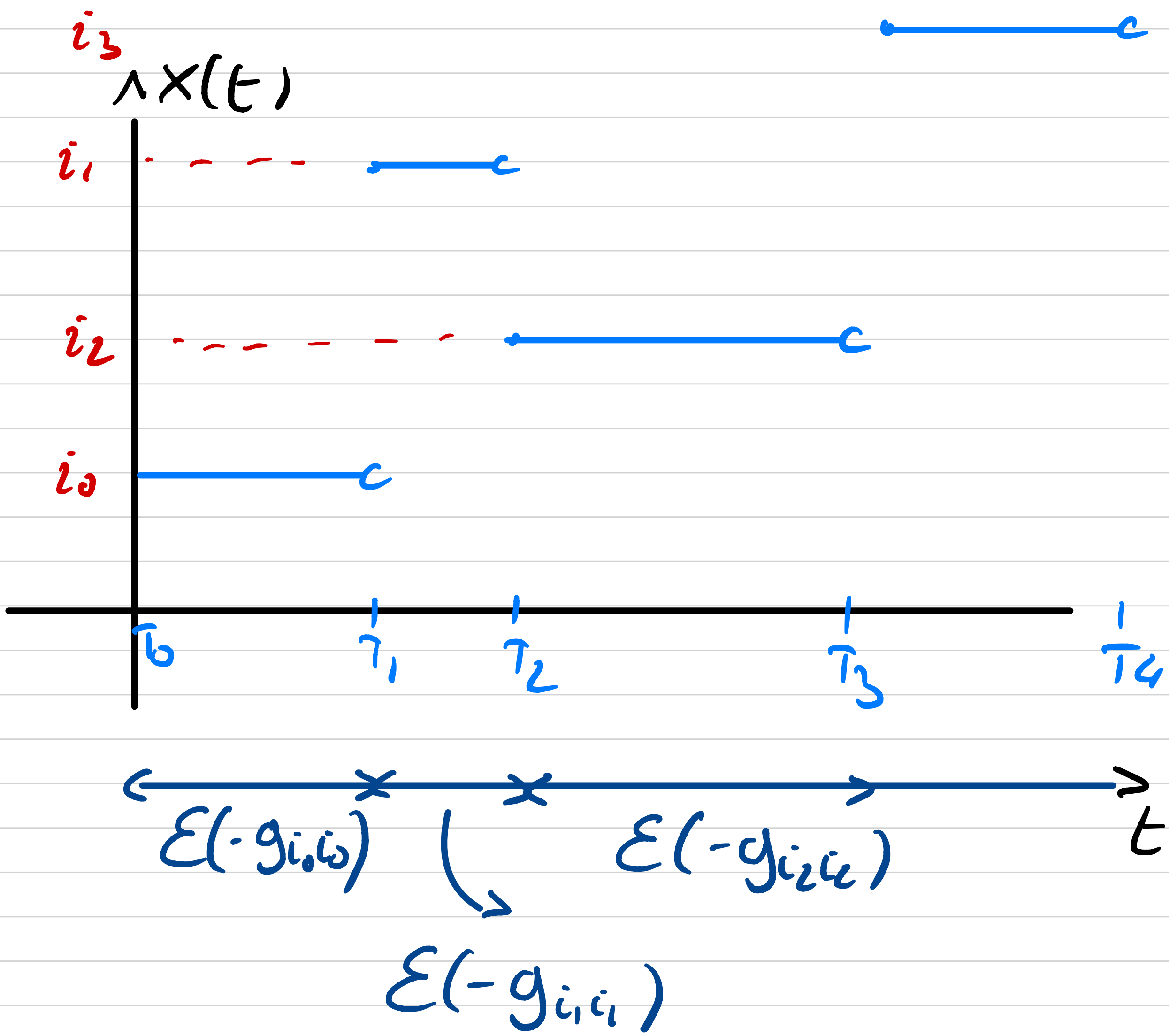
$X \sim \mathcal{N}(\mu, \sigma^2)$

density of X

Then

$$\mathcal{L}(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \, dx$$

Graph



$x(t)$
i_3
i_1
i_2
i_0
T_0
T_1
T_2
T_3
T_4
t
$\mathcal{E}(-g_{i_0 i_0})$
$\mathcal{E}(-g_{i_2 i_2})$
$\mathcal{E}(-g_{i_1 i_1})$

Recall An r.v $X \sim \mathcal{E}(\lambda)$ represents a non aging system

$$\underbrace{P(X > a+b \mid X > a)}_{\text{probab. to live} > b \text{ years when our age is } a} = \underbrace{P(X > b)}_{\text{probab to live} > b \text{ years as a baby}}$$

Rmk We also have

$$\left.\begin{array}{l} \text{If } X \text{ has a density} \\ + \; X \text{ not aging} \end{array}\right\} \Rightarrow X \sim \mathcal{E}(\lambda) \text{ for some } \lambda$$