Generating function for a sum

Theorem 6.

Let

- X, Y random variables
- $X \perp\!\!\!\perp Y$

Then

$$G_{X+Y}(s) = G_X(s)G_Y(s)$$

Proof that $G_Z(s) = G_X(s)\ G_Y(s)$

We have $X \perp\!\!\!\perp Y$ and $Z = X+Y$

First method : $f_Z = f_X * f_Y$

$$\Rightarrow G_Z(s) = G_X(s)\ G_Y(s)$$

Second method :

$$G_Z(s) = E[s^Z] = E[s^{X+Y}]$$

$$= E[s^X s^Y] \overset{X \perp\!\!\!\perp Y}{=} E[s^X]\ E[s^Y]$$

$$\Rightarrow \boxed{G_Z(s) = G_X(s)\ G_Y(s)}$$

Binomial random variables

Let $X \sim \text{Bin}(n, p)$ $\quad n \geq 1 \quad p \in (0,1)$

Then

$$P(X = k) = \binom{n}{k} p^k q^{n-k}, \quad q = 1-p$$

Claim :

$$\boxed{G_X(s) = [(1-p) + ps]^n}$$

Main use of $Bin(n, p)$ r.v:

\# successes in n $\perp\!\!\!\perp$ trials with success with proba p

Another way to express that:

Let $Y_1, \dots, Y_n$ be n $\perp\!\!\!\perp$ random variables with $Y_j \sim B(p)$

Set $X = \sum_{j=1}^{n} Y_j = Y_1 + Y_2 + \dots + Y_n$.

Then

$$X \sim Bin(n, p)$$

Generating function

$$G_X(s) = \prod_{j=1}^{n} \overbrace{G_{Y_j}(s)}^{(1-p)+ps} = ((1-p)+ps)^n$$

Binomial random variable (1)

Notation:

$$X \sim \mathrm{Bin}(n, p), \text{ for } n \geq 1,\ p \in (0,1)$$

State space:

$$\{0, 1, \ldots, n\}$$

Pmf:

$$\mathbf{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n$$

Expected value, variance and generating function:

$$\mathbf{E}[X] = np, \qquad \mathbf{Var}(X) = np(1-p), \qquad G_X(s) = [(1-p) + p\,s]^n$$

Binomial random variable (2)

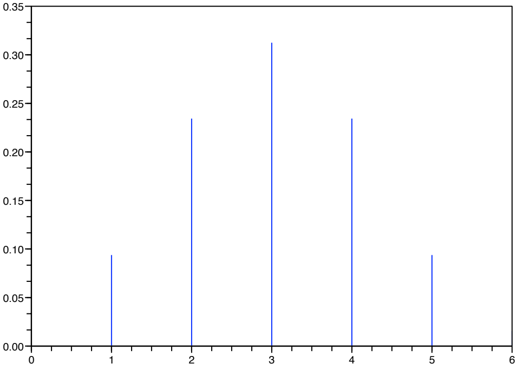
Use 1, Number of successes in a Bernoulli trial:

- Example: Roll a dice 9 times.
- $X = \#$ of 3 obtained
- We get $X \sim \text{Bin}(9, 1/6)$
- $\mathbf{P}(X = 2) = 0.28$ $= \binom{9}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^7$

Use 2: Counting a feature in a repeated trial:

- Example: stock of 1000 pants with 10% defects
- Draw 15 times a pant at random
- $X = \#$ of pants with a defect
- We get $X \sim \text{Bin}(15, 1/10)$

Binomial random variable (3)



$x \in \{0, 1, .., 6\}$

Figure: Pmf for $\text{Bin}(6; 0.5)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Binomial random variable (4)

If n large $\quad Bin(n, p) \approx \mathcal{N}(np, np(1-p))$

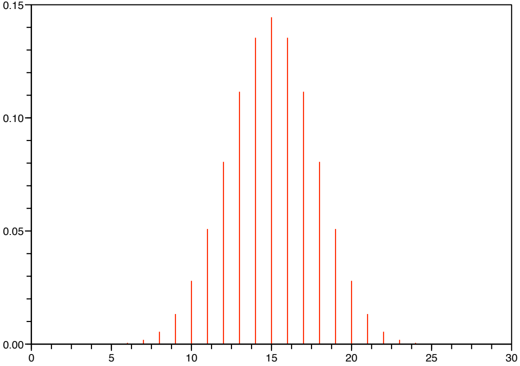


Figure: Pmf for Bin(30; 0.5). x-axis: k. y-axis: $\mathbf{P}(X = k)$

Computation for G_X

Generating function for Bernoulli: If $Y \sim \mathcal{B}(p)$ then

$$G_Y(s) = (1 - p) + p\, s$$

Decomposition of Binomial: If $X \sim \text{Bin}(n, p)$ one can write

$$X = \sum_{i=1}^{n} Y_i, \quad \text{with} \quad Y_i \text{ i.i.d}, \; Y_i \sim \mathcal{B}(p)$$

Computing G_X: We get

$$G_X(s) = \prod_{i=1}^{n} G_{Y_i}(s) = [(1 - p) + p\, s]^n$$

Joint generating functions

Definition 7.

Let

- X_1, X_2 random variables
- X_1, X_2 take values in $\mathbb{Z}$

Then the pgf for (X_1, X_2) is

$$G_{X_1,X_2}(s_1, s_2) = \mathbf{E}\left[s_1^{X_1} s_2^{X_2}\right]$$

Characterization of independence

Theorem 8.

Let

- X_1, X_2 random variables
- G_{X_1,X_2} the corresponding pgf

Then we have

$$X_1 \perp\!\!\!\perp X_2 \quad \Longleftrightarrow \quad G_{X_1,X_2}(s_1, s_2) = G_{X_1}(s_1) G_{X_2}(s_2) \text{ for all } s_1, s_2$$

Outline

Definition of random walk

Definition 9.

Let

- $X_1, \ldots, X_n$ ~~Bernoulli~~ random variables with values ± 1,

$$\mathbf{P}(X_i = 1) = p, \qquad \mathbf{P}(X_i = -1) = 1 - p = q$$

- The X_i's are independent

Idea: at each step, we flip a coin to know if we are doing one step up or down

We set $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i$$

Then S is called simple random walk

Symmetric random walk

Definition 10.

Let

- $X_1, \ldots, X_n$ Ber~~n~~oulli random variables with values ± 1,

$$\mathbf{P}(X_i = 1) = \frac{1}{2}, \qquad \mathbf{P}(X_i = -1) = \frac{1}{2}$$

- The X_i's are independent

We set $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i$$

Then S is called symmetric random walk

Illustration: 30 steps of a random walk

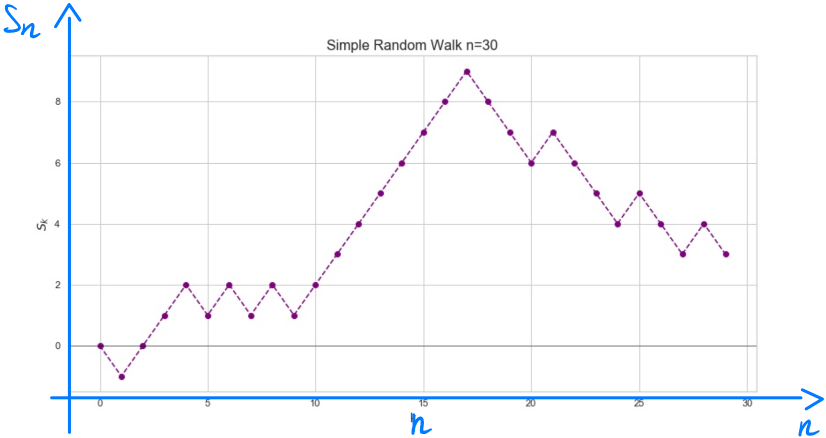
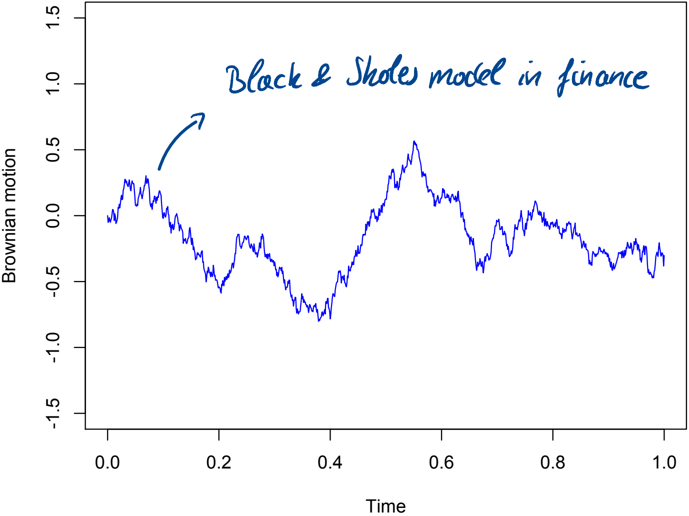


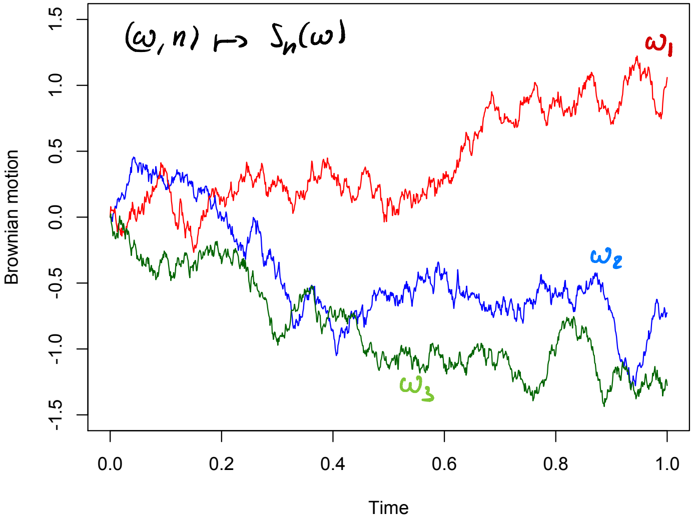
Illustration: chaotic path (Brownian motion)

Take S_n, accelerate time, squeeze space $\longrightarrow$ Brownian motion



Black & Sholes model in finance

Illustration: random path (Brownian motion)



Questions about random walks

Main questions

1. Does the walk S_n go to ∞ when $n \to \infty$?
2. Does it return to 0 after $n = 0$?
3. How often does it return to 0?
4. What is the range of $S_n(\omega)$?

Methodologies to answer those questions

1. Elementary methods based on generating functions
2. Later: Markov chain methods
3. Also useful: martingale methods

If X is a random variable, e.g. $X \sim Bin(n,p)$

Then $X : \Omega \longrightarrow \{0, 1, \dots, n\}$

and we should write $X(\omega)$

Event : any $A \subset \Omega$

Ex 1 For the Bin, one can take $\Omega = \{0,1\}^n$

we have $\omega \in \Omega$. For each $\omega \in \Omega$,

we get a value $X(\omega)$

Ex 2 For the rw from 0 to n, one can take

$$\Omega = \{-1, 1\}^n$$