Proof of Proposition 31

Properties of exponential random variables: If $Z \sim \mathcal{E}(\mu)$, then

$$\mathbf{P}\left(Z > a + b \,|\, \cancel{X}\, Z > a\right) = \mathbf{P}\left(Z > b\right) = \exp\left(-\mu\, b\right) \qquad (7)$$

Remarks about (7):

1. Relation (7) can be interpreted as lack of memory
2. It can also be interpreted as no aging
3. In fact (7) characterizes the distribution $\mathcal{E}(\mu)$

$U = \inf\{t \geq 0;\ X(s+t) \neq i\}$

Proof We are going to prove

$$\mathbb{P}(U > a+b \mid U > a,\ X(s) = i)$$

$$= \mathbb{P}(U > b \mid X(s) = i)$$

We have

we are still at i at time $s+a$

$X(s+a) = i$

$$\mathbb{P}(U > a+b \mid \overbrace{U > a}^{}\ ,\ X(s) = i)$$

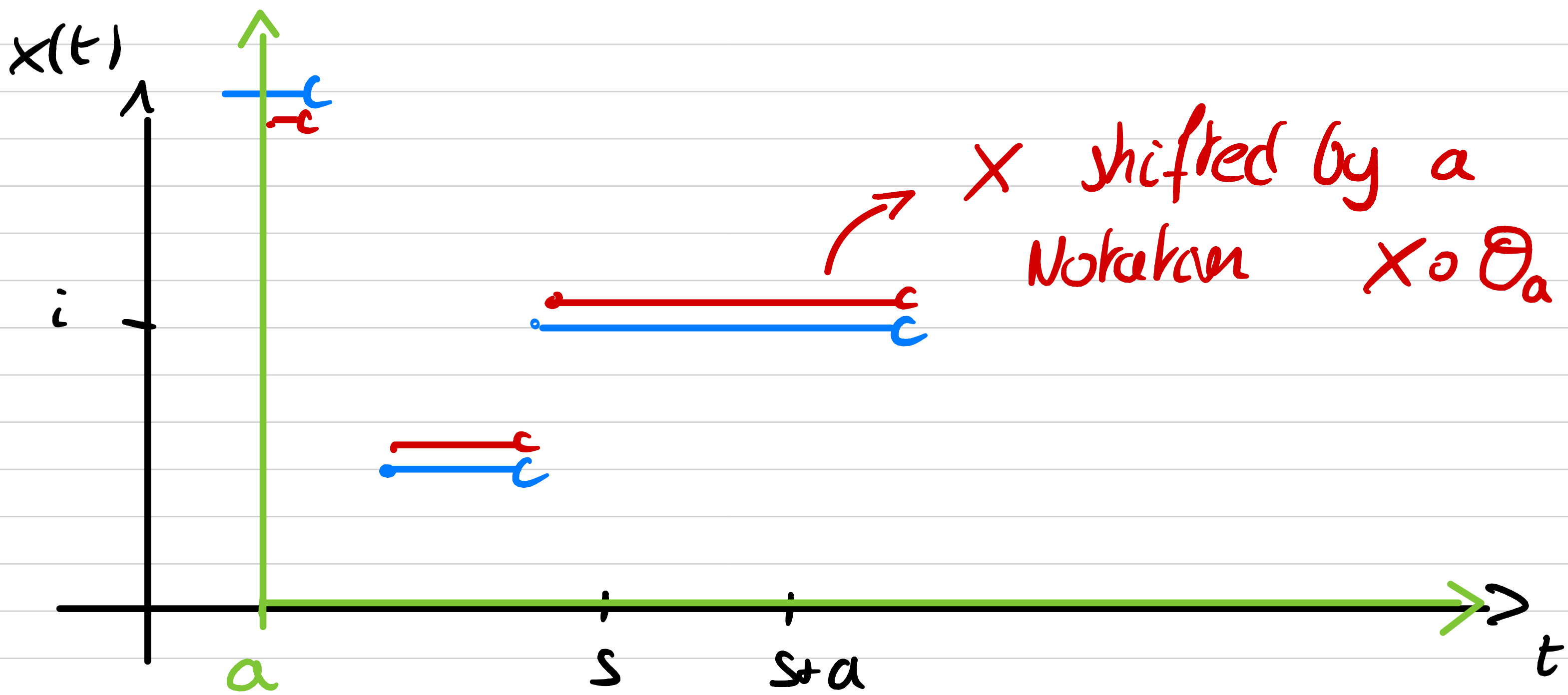
$$= \mathbb{P}(U > a+b \mid X(s+a) = i,\ X(s) = i)$$

$$= \mathbb{P}(U > a+b \mid X(s+a) = i)$$

$$\boxed{\mathbb{P}(U > a+b \mid U > a, X(s)=i)}$$

$$= \mathbb{P}(U > a+b \mid X(s+a)=i, X(s)=i)$$

$$= \mathbb{P}(U > a+b \mid X(s+a)=i)$$



$$= \mathbb{P}(U\circ\theta_a > b \mid [X\circ\theta_a](s) = i)$$

↓ X homogeneous

$$\boxed{= \mathbb{P}(U > b \mid X(s) = i)}$$

Proof of Proposition 31

Main argument: We have

$$
\begin{aligned}
&\mathbf{P}\left(U > a+b \middle| U > a,\, X(s) = i\right) \\
&= \mathbf{P}\left(U > a+b \middle| X(s+a) = i,\, X(s) = i\right) \\
&= \mathbf{P}\left(a + U \circ \theta_a > a+b \middle| X(s+a) = i,\, X(s) = i\right) \\
&= \mathbf{P}\left(U \circ \theta_a > b \middle| X(s+a) = i\right) \qquad \text{(Markov)} \\
&= \mathbf{P}\left(U \circ \theta_a > b \middle| X(s) \circ \theta_a = i\right) \\
&= \mathbf{P}\left(U > b\right) \qquad \text{(Homogeneity)}
\end{aligned}
$$

Imbedded Markov chain

Proposition 32.

Let

- X Markov chain with standard transition P_t
- Assume $X(0) = i$

Then we have

$$\mathbf{P}\left(X \text{ jumps to } j \,|\, X(0) = i\right) = -\frac{g_{ij}}{g_{ii}}$$

$$\sum_{j \neq i} g_{ij} = -g_{ii} \implies -\sum_{j \neq i} \frac{g_{ij}}{g_{ii}} = 1$$

$$\implies \left(-\frac{g_{ij}}{g_{ii}}\right)_{i \in S, j \neq i} \text{ transition matrix for a Markov chain}$$

Idea of the proof On a small interval $[t, t+h)$, if $j \neq i$

$$\boxed{\mathbb{P}(X \text{ jumps to } j \,,\, X \text{ jumps} \mid X(t) = i)}$$

$$= \frac{\mathbb{P}(X \text{ jumps to } j,\ X(t) = i)}{\underbrace{\mathbb{P}(X \text{ jumps} \mid X(t) = i)}}$$

$\hookrightarrow$ $1 - \mathbb{P}(X \text{ stays put} \mid X(t) = i)$

$$\simeq \frac{g_{ij}\, h}{1 - \mathbb{P}(X(t+h) = i \mid X(t) = i)}$$

$$\simeq \frac{g_{ij}\, h}{1 - (1 + g_{ii}\, h)}$$

$$= -\frac{g_{ij} h}{g_{ii} h} \qquad \boxed{= -\frac{g_{ij}}{g_{ii}}}$$

Proof of Proposition 32

Argument on a small interval: On $[t, t+h)$,

$$\begin{aligned} \mathbf{P}\left(X \text{ jumps to } j \,|\, X \text{ jumps}\right) &\simeq \frac{p_{ij}(h)}{1 - p_{ii}(h)} \\ &\simeq \frac{g_{ij}\, h}{(-g_{ii}\, h)} \\ &\simeq -\frac{g_{ij}}{g_{ii}} \end{aligned}$$

Example with 2 states (1)

Model: We consider

- State space $S = \{1, 2\}$
- Generator

$$G = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

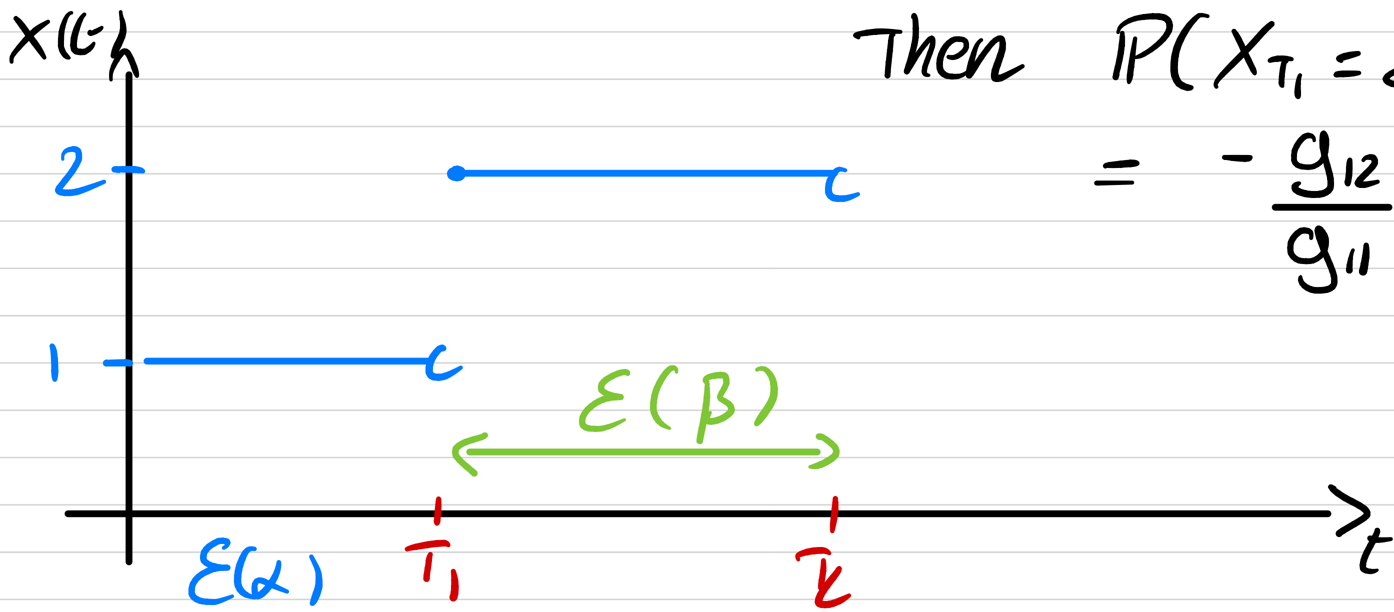
$S = \{1, 2\}$

According to an decomposition

$$G = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

If $X(t) = 1$, then

$$T_1 \equiv \inf\{ s ; X(t+s) \neq 1 \} \sim \mathcal{E}(\underbrace{\alpha}_{-g_{11}})$$



Then $P(X_{T_1} = 2 \mid X(0) = 1)$

$$= -\frac{g_{12}}{g_{11}} = \frac{\alpha}{\alpha} = 1$$

Example with 2 states (2)

Pathwise description: Applying Propositions 31 and 32 we get

1. If X is in state 1 then
   - X stays at 1 an amount of time $\sim \mathcal{E}(\alpha)$
   - Next X jumps to 2
2. If X is in state 2 then
   - X stays at 2 an amount of time $\sim \mathcal{E}(\beta)$
   - Next X jumps to 1

Next step

① Can we compute $p_{ij}(t)$ $\forall t, \forall i,j \in \{1,2\}$?

② Can we compute

$$\lim_{t \to \infty} p_{ij}(t) \quad ?$$

③ Invariant measure

Main tool: use

$$P_t' = G P_t$$

$$P_t = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix}$$

$$G \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

Example with 2 states (3)

Forward equation: Can be read as

$$\begin{bmatrix} p'_{11}(t) & p'_{12}(t) \\ p'_{21}(t) & p'_{22}(t) \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

Sub-system for p_{11}, p_{12}: We get a separate system of the form

$$\begin{bmatrix} p'_{11}(t) \\ p'_{12}(t) \end{bmatrix} = A \begin{bmatrix} p_{11}(t) \\ p_{12}(t) \end{bmatrix}, \quad \text{with} \quad A = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix} = G^T$$

Example with 2 states (3)

Eigenvalue decomposition for A: We get

$$\lambda_1 = 0, \quad \text{with} \quad \mathbf{v}_1 = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

$$\lambda_2 = -(\alpha + \beta), \quad \text{with} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

gen solution: $c_1 e^{-\lambda_1 t} v_1 + c_2 e^{-\lambda_2 t} v_2$

General form of the solution: We get

$$\begin{bmatrix} p_{11}(t) \\ p_{12}(t) \end{bmatrix} = c_1 \begin{bmatrix} \beta \\ \alpha \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp\left(-(\alpha + \beta) t\right)$$

Example with 2 states (4)

Computation of constants: We use

$$\lim_{t\to\infty} (p_{11}(t) + p_{12}(t)) = 1, \quad \text{and} \quad p_{12}(0) = 0$$

and we get

$$c_1 = \frac{1}{\alpha+\beta}, \quad \text{and} \quad c_2 = -\frac{\alpha}{\alpha+\beta}$$

Unique solution: We end up with

$$\begin{bmatrix} p_{11}(t) \\ p_{12}(t) \end{bmatrix} = \frac{1}{\alpha+\beta}\begin{bmatrix} \beta \\ \alpha \end{bmatrix} - \frac{\alpha}{\alpha+\beta}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp\left(-(\alpha+\beta)t\right)$$

$$\xrightarrow{t\to\infty} \begin{pmatrix} \beta/\alpha+\beta \\ \alpha/\alpha+\beta \end{pmatrix}$$ invariant measure

Example with 2 states (5)

Sub-system for p_{21}, p_{22}: We get a separate system of the form

$$\begin{bmatrix} p'_{21}(t) \\ p'_{22}(t) \end{bmatrix} = A \begin{bmatrix} p_{21}(t) \\ p_{22}(t) \end{bmatrix}, \quad \text{with} \quad A = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}$$

Unique solution: We end up with

$$\begin{bmatrix} p_{21}(t) \\ p_{22}(t) \end{bmatrix} = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} - \frac{\beta}{\alpha+\beta} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp\left(-(\alpha+\beta)t\right)$$