Outline

Irreducibility of chains

Proposition 33.

Let X Markov chain with standard transition P_t

Then we have

from i, one can never reach j (should be seen on g(x))

1. For every pair $i, j \in S$, either
$$p_{ij}(t) = 0 \text{ for all } t > 0$$
or
$$p_{ij}(t) > 0 \text{ for all } t > 0$$
2. Terminology: if $p_{ij}(t) > 0$ for all $t > 0$
$\hookrightarrow$ X is said to be irreducible
3. In order to know if X is irreducible
$\hookrightarrow$ draw graph related to G

Birth process example

Recall: For the birth process,

$$G = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ 0 & -\lambda_1 & \lambda_1 & 0 & 0 & \cdots \\ 0 & 0 & -\lambda_2 & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Nature of states:

All states are transient

Graph of X

$$\textcircled{0} \to \textcircled{1} \to \textcircled{2} \to \textcircled{3} \to$$

There is no closed class

$\Rightarrow$ every state is transient

Rmk we already know that

$$\lim_{t \to \infty} N(t) = \infty$$

$\Rightarrow$ $\boxed{\text{No invariant measure}}$

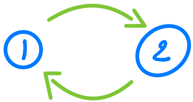
2 states example

Recall:

$$G = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

Nature of states:

The chain is irreducible



we have a unique class

$\Rightarrow$ X irreducible

$\Rightarrow$ $p_{ij}(t) > 0 \quad \forall\, i,j \quad \forall t > 0$

Rmk By solving $P_t' = P_t G$, we have seen

$$\begin{pmatrix} p_{11}(t) \\ p_{12}(t) \end{pmatrix} = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta \\ \alpha \end{pmatrix} - \frac{\alpha}{\alpha+\beta}\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-(\alpha+\beta)t}$$

In particular, one can easily see

$$\boxed{p_{11}(t) > 0 \quad , \quad p_{12}(t) > 0 \quad \forall t > 0}$$

Same thing for $p_{21}(t)$, $p_{22}(t)$

Stationary distribution

Definition 34.

Let

- X Markov chain with transition P_t
- π vector

Then π is a stationary distribution if

1. $\pi_j \geq 0$ for all $j \in S$ and $\sum_{j \in S} \pi_j = 1$
2. π satisfies $\pi = \pi P_t$ for all $t \geq 0$, that is

$$\pi_j = \sum_{i \in S} \pi_i p_{ij}(t), \quad \text{for all} \quad j \in S, \ \forall t \geq 0$$

$\boxed{Q:}$ Can we make those conditions simpler?

Rmk If $\pi P_t = \pi$ for all t, then

$$\pi P_t - \pi \, \mathrm{Id} = 0$$

$$\Rightarrow \pi \frac{(P_t - \mathrm{Id})}{t} = \frac{0}{t}$$

As $t \to 0$ we get

$$\boxed{\pi G = 0}$$

Interpretation of stationary distribution

Proposition 35.

Let

- X Markov chain with transition P
- π invariant distribution

Then

$$\overset{X(0)}{\cancel{X}_0} \sim \pi \quad \Longrightarrow \quad X(t) \sim \pi \quad \text{for all } t \geq 0$$

Otherwise stated,

$$\mathbf{P}\left(X(t) = j \,|\, X(0) \sim \pi\right) = \pi_j$$

Stationary distribution and generator

Proposition 36.

Let

- X Markov chain with transition P and generator G
- π distribution

Then

$$\pi \text{ invariant distribution} \quad \overset{\Longrightarrow}{\Longleftrightarrow} \quad \pi G = 0$$

Proof of Proposition 36

Basic relation: We have

$$\pi G = 0 \quad \Longleftrightarrow \quad \pi G^n = 0$$

Reasoning with matrix exponential: We get

$$\begin{aligned}
\pi G = 0 \quad &\Longleftrightarrow \quad \sum_{n=1}^{\infty} \frac{t^n}{n!} \pi G^n = 0, \quad \text{for all } t \geq 0 \\
&\Longleftrightarrow \quad \pi \sum_{n=1}^{\infty} \frac{t^n}{n!} G^n = 0, \quad \text{for all } t \geq 0 \\
&\Longleftrightarrow \quad \pi \sum_{n=0}^{\infty} \frac{t^n}{n!} G^n = \pi, \quad \text{for all } t \geq 0 \\
&\Longleftrightarrow \quad \pi P_t = \pi, \quad \text{for all } t \geq 0
\end{aligned}$$

Ergodic theorem

Proposition 37.

Let

- X Markov chain with transition P and generator G
- Assume X is irreducible

Then

1. If there exists a stationary distribution π, then
$$\pi \text{ is unique and } \lim_{t\to\infty} p_{ij}(t) = \pi_j \text{ for all } i, j \in S$$
2. If there is no stationary distribution π, then
$$\lim_{t\to\infty} p_{ij}(t) = 0 \text{ for all } i, j \in S$$

2 states example (1)

Recall:

$$G = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

Invariant distribution: The chain is irreducible and we have

$$\pi = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix} \implies \pi G = 0$$

$\Rightarrow \quad p_{ij}(t) \longrightarrow \pi_j \quad$ for $i = 1, 2$

(ergodic theorem)

2 states example (2)

$$G = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

Recall: We have seen

$$\begin{bmatrix} p_{11}(t) \\ p_{12}(t) \end{bmatrix} = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} - \frac{\alpha}{\alpha+\beta} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp\left(-(\alpha+\beta)t\right)$$

Verifying the ergodic theorem: We get

$$\lim_{t\to\infty} \begin{bmatrix} p_{11}(t) \\ p_{12}(t) \end{bmatrix} = \underbrace{\frac{1}{\alpha+\beta} \begin{bmatrix} \beta \\ \alpha \end{bmatrix}}_{\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}} = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}$$

Ergodic theorem is satisfied on this simple example