Problems for Final Spring 24

6.5.**4.** Can a reversible chain be periodic?

Period 2 The Markov chain on $S = \{1, 2\}$ with transition

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is reversible with

$$\pi = \left(\tfrac{1}{2} \quad \tfrac{1}{2} \right)$$

It has period $d = 2$

Period 3 If X with transition P has period 3, it means that

(i) $p_{i_1 i_2} \, p_{i_2 i_1} = 0 \quad \forall \, i_1, i_2$

(ii) For all $i_1 \in S$, $\exists \, i_2, i_3$ s.t

$$p_{i_1 i_2} \, p_{i_2 i_3} \, p_{i_3 i_1} > 0$$

Using detailed balance consider three states called i_1, i_2, i_3.
We can assume $\pi_i > 0$ for all i's.
Then

$$\pi_{i_1} P_{i_1 i_2} P_{i_2 i_3} P_{i_3 i_1}$$

$$= P_{i_2 i_1} \pi_2 P_{i_2 i_3} P_{i_3 i_1}$$

$$= P_{i_2 i_1} P_{i_3 i_2} \pi_{i_3} P_{i_3 i_1}$$

$$= P_{i_2 i_1} P_{i_3 i_2} P_{i_1 i_3} \pi_{i_1}$$

In fact we have proved that for a reversible MC and every i_1, i_2, i_3 we have

$$\boxed{P_{i_1 i_2} P_{i_2 i_3} P_{i_3 i_1} = P_{i_1 i_3} P_{i_3 i_2} P_{i_2 i_1}}$$

contradiction Assume now that $d=3$ and consider i_1, i_2, i_3 s.t.

$$p_{i_1 i_2} \; p_{i_2 i_3} \; p_{i_3 i_1} > 0 \qquad (1)$$

we also have

$$p_{i_1 i_2} \; p_{i_2 i_3} \; p_{i_3 i_1} = p_{i_1 i_3} \; p_{i_3 i_2} \; p_{i_2 i_1} \qquad (2)$$

Now if $d=3$, for all i_1, i_2 we have

$$p_{i_1 i_2} \; p_{i_2 i_1} = 0$$

Hence $p_{i_1 i_2} = 0$ or $p_{i_2 i_1} = 0$.
Thus in relation (2) we have

$$p_{i_1 i_2} \; p_{i_2 i_3} \; p_{i_3 i_1} = p_{i_1 i_3} \; p_{i_3 i_2} \; p_{i_2 i_1} = 0$$

This contradicts (1). Hence

$$\boxed{d \text{ cannot be } = 3}$$

Case $d \geq 4$ Easy generalization of the case $d=3$, using the cycle property

$$p_{i_1 i_2} \, p_{i_2 i_3} \cdots p_{i_{n-1} i_n}$$

$$= p_{i_1 i_n} \, p_{i_n i_{n-1}} \cdots p_{i_2 i_1}$$

6.5.6. Which of the following (when stationary) are reversible Markov chains?

(a) The chain $X = \{X_n\}$ having transition matrix $\mathbf{P} = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$ where $\alpha + \beta > 0$.

General strategy One can either

(i) Compute the invariant π by solving $\pi P = \pi$, then check detailed balance $\pi_i P_{ij} = \pi_j P_{ji}$

(ii) Directly find π verifying detailed balance.

Finding an invariant π Note that

$$\pi P = \pi \iff P^t \pi^t = \pi^t$$

$$\iff (P^t - Id)\, \pi^t = 0$$

This system looks more like your usual linear system of equations.

Application of (i) we want to solve

$$(P^t - I)\pi^t = 0$$

$$\Leftrightarrow \begin{pmatrix} -\alpha & \beta \\ \alpha & -\beta \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = 0$$

Hence $\begin{cases} \beta \pi_2 = \alpha \pi_1 \\ \pi_1 + \pi_2 = 1 \end{cases}$

$$\Rightarrow \quad \pi_1 = \frac{\beta}{\alpha+\beta}, \quad \pi_2 = \frac{\alpha}{\alpha+\beta}$$

Then we easily get

$$\pi_1 P_{12} = \frac{\beta}{\alpha+\beta} \cdot \alpha$$

$$= \pi_2 P_{21}$$

Hence $\boxed{X \text{ is reversible}}$

Application of (ii) The detailed balance is written as

$$\pi_1 p_{12} = \pi_2 p_{21}$$

$$\iff \pi_1 \alpha = \pi_2 \beta$$

we thus get the same system as before:

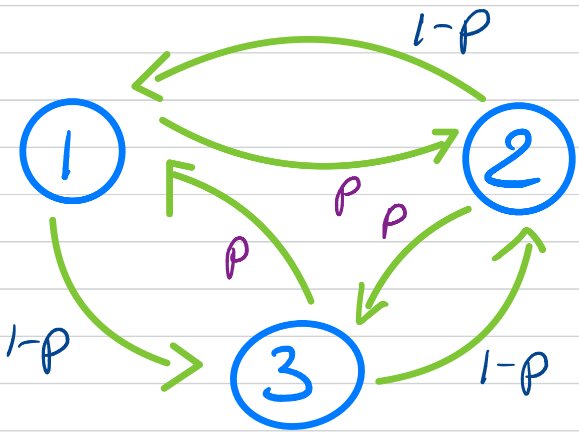
$$\begin{cases} \alpha \pi_1 = \beta \pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

we get that

$\boxed{X \text{ is reversible}}$

(b) The chain $Y = \{Y_n\}$ having transition matrix $\mathbf{P} = \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$ where $0 < p < 1$.

Graph of Y



Intuition: by symmetry, the invariant measure is

$$\boxed{\pi = \left(\tfrac{1}{3} \quad \tfrac{1}{3} \quad \tfrac{1}{3}\right)}$$

Checking invariant measure One easily checks

$$\pi P = \pi$$

checking detailed balance The detailed balance has to be written for $\pi = \frac{1}{3}(1 \quad 1 \quad 1)$. Then we get

$$\pi_1 p_{12} = \pi_2 p_{21} \iff p_{12} = p_{21}$$

$$\iff p = 1 - p \iff p = \tfrac{1}{2}$$

Thus $\boxed{p \neq \tfrac{1}{2} \Rightarrow Y \text{ not reversible}}$

Case $p=\frac{1}{2}$ we have

$\boxed{p=\frac{1}{2}}$ $\Rightarrow$ P symmetric

$\boxed{\Rightarrow \quad Y \text{ reversible}}$

6.8.4. Let B be a simple birth process (6.8.11b) with $B(0) = I$; the birth rates are $\lambda_n = n\lambda$. Write down the forward system of equations for the process and deduce that

$$\mathbb{P}\big(B(t) = k\big) = \binom{k-1}{I-1} e^{-I\lambda t}\big(1 - e^{-\lambda t}\big)^{k-I}, \qquad k \geq I.$$

Show also that $\mathbb{E}(B(t)) = Ie^{\lambda t}$ and $\mathrm{var}(B(t)) = Ie^{2\lambda t}(1 - e^{-\lambda t})$.

Generator The general expression for the generator is

$$G = \begin{pmatrix} -\lambda_0 & \lambda_0 & & 0 \\ & \ddots & \ddots & \\ 0 & & -\lambda_n & \lambda_n \\ & & & \ddots \quad \ddots \end{pmatrix}$$

For the simple birth process we remove the trivial $\lambda_0 = 0$ and

$$G = \begin{pmatrix} -\lambda & \lambda & & 0 \\ & -2\lambda & 2\lambda & \\ 0 & & \ddots & \ddots \end{pmatrix}$$

Forward equation written as $P'_t = P_t G$ or

$$p'_{ij}(t) = \sum_{k\in\mathbb{N}^*} p_{ik}(t)\, g_{kj} \qquad (1)$$

we have

$$g_{j-1,j} = (j-1)\lambda \quad , \quad g_{jj} = -j\lambda$$

and $g_{kj} = 0$ otherwise. Hence equation (1) becomes

$$p'_{ij} = (j-1)\lambda\, p_{i,j-1} - j\lambda\, p_{ij} \qquad (2)$$

Generating function For a given i we set

$$G_i(s,t) = \sum_{j\in\mathbb{N}^*} p_{ij}(t)\, s^j \qquad (3)$$

Plugging (2) into (3) we get

$$\partial_t G_i(s,t) = \sum_{j\in\mathbb{N}} \left(\lambda(j-1)\, p_{i,j-1} - \lambda j\, p_{ij}\right) s^j$$

Differential equation we have obtained

$$\partial_t G_i(s,t) = \sum_{j\in\mathbb{N}} \left(\lambda (j-1)\, p_{i,j-1} - \lambda j\, p_{ij}\right) s^j$$

$$= \lambda s \sum_{j\in N^*} (j-1)\, p_{i,j-1}\, s^{j-1}$$

$$- \lambda \sum_{j\in\mathbb{N}} j\, p_{ij}\, s^j$$

$$= \lambda (s-1)\, s \sum_{j\in\mathbb{N}^*} j\, p_{ij}\, s^{j-1}$$

$$= \lambda (s-1) s\, \partial_s G_i(s,t)$$

Hence G_i satisfies the pde

$$\begin{cases} \partial_t G_i(s,t) = \lambda (s-1) s\, \partial_s G_i(s,t) \\ G_i(s,0) = s^i \end{cases}$$

1st order linear PDE (Evans).

$$\begin{cases} b(x)\cdot Du + c(x)\, u = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \quad (4)$$

Eq: $b(x)\cdot Du + c(x)u = 0$, $u|_{\partial U} = g$

Our case For our pgf we have

$$\begin{cases} \lambda(s-1)s\, \partial_s G_i - \partial_t G_i = 0, \\ \underbrace{G_i(s,0)}_{\partial U} = \underbrace{s^i}_{g} \end{cases}$$

so that refering to (4) we get

$$b(s,t) = \begin{pmatrix} \lambda(s-1)s \\ -1 \end{pmatrix}, \qquad c = 0$$

Characteristic curve of the form

$$(s(\sigma), t(\sigma)) \equiv \vec{x}(\sigma).$$

Then we set

$$z(\sigma) = G_i(\vec{x}(\sigma))$$

we want z to solve a 1st order ode.

Equation for the characteristics In Evans p. 100, the linear case is reduced to a couple of ode's for the characteristics:

$$\vec{x}'(\sigma) = b(\vec{x}(\sigma)), \quad z'(\sigma) = -c(\vec{x}(\sigma))z(\sigma),$$

which can be expressed for G_i as

$$\vec{x}'(\sigma) = \begin{pmatrix} s'(\sigma) \\ t'(\sigma) \end{pmatrix} = \begin{pmatrix} \alpha \, s(\sigma)(s(\sigma)-1) \\ -1 \end{pmatrix}$$

and

$$z'(\sigma) = 0 \implies \overbrace{G_i(\vec{x}(\sigma))}^{\text{gives the value of } G_i} = C$$

Interpretation G_i is constant along $(\vec{x}(\sigma))_\sigma$ and we have to find a proper $(\vec{x}(\sigma))_\sigma$ s.t. $\vec{x}(0) = (s_0, 0)$ and $\vec{x}(\sigma_0) = (s, t)$ for a given σ_0

Solving for x we have

Recall

$$\begin{pmatrix} s'(\sigma) \\ t'(\sigma) \end{pmatrix} = \begin{pmatrix} \alpha\, s(\sigma)(s(\sigma)-1) \\ -1 \end{pmatrix}$$

(i) $$\frac{ds}{s(s-1)} = \alpha\, d\sigma$$

$$\frac{1}{s-1} - \frac{1}{s} = \alpha\, d\sigma$$

$$\ln\left|\frac{s-1}{s}\right| = \alpha\sigma + c_1$$

$$\frac{s-1}{s} = c_2 e^{\alpha\sigma}$$

Note: here $c_1, c_2 \in \mathbb{R}$

$$\frac{s-1}{s} = a \iff s-1 = as$$

$$\iff (1-a)s = 1$$

$$\iff s = \frac{1}{1-a}$$

$$s = \frac{1}{1-c_2 e^{\alpha\sigma}}$$

(ii) $$t'(\sigma) = -1 \implies t = c_3 - \sigma$$

Hence $$\bar{x}'(\sigma) = \begin{pmatrix} \dfrac{1}{1-c_2 e^{\alpha\sigma}} \\ c_3 - \sigma \end{pmatrix}$$

$\partial_t G_i(s,t) = \lambda(s-1)s\,\partial_s G_i(s,t)$ $\qquad G(s,0) = s^i$

Initial condition for $\vec{x}$ we have found

$$\vec{x}(\sigma) = \begin{pmatrix} \dfrac{1}{1 - c_2 e^{\lambda\sigma}} \\[2ex] c_3 - \sigma \end{pmatrix}$$

At time $\sigma = 0$ we want to be on the boundary $\{s \in [0,1],\ t = 0\}$, so that $c_3 = 0$. Therefore

$$\begin{cases} \vec{x}(\sigma) = \begin{pmatrix} \dfrac{1}{1 + c_4 e^{\lambda\sigma}} \\[2ex] -\sigma \end{pmatrix} \\[4ex] G_i(\vec{x}(\sigma)) = G_i(\vec{x}(0)) = \left(\dfrac{1}{1+c_4}\right)^i \end{cases}$$

G_i constant along $\vec{x}(\sigma)$

$\bar{x}'(\sigma) = \begin{pmatrix} \frac{1}{1+c_4 e^{\lambda\sigma}} \\ -\sigma \end{pmatrix}$ $\qquad u(\bar{x}'(\sigma)) = u(\bar{x}'(0)) = \left(\frac{1}{1+c_4}\right)^i$

Value at s Take $s \in [0,1]$, $t \geq 0$ and invert : $\sigma = -t$ and

$$\frac{1}{1+c_4 e^{-\lambda t}} = s \iff \frac{1}{s} = 1 + c_4 e^{-\lambda t}$$

$$\iff c_4 e^{-\lambda t} = \frac{1-s}{s}$$

$$\iff c_4 = \frac{1-s}{s} e^{\lambda t}$$

$$\iff 1 + c_4 = \frac{(1-s)e^{\lambda t} + s}{s}$$

$$= \frac{(1-s) + s e^{-\lambda t}}{s e^{-\lambda t}}$$

$$= \frac{1 - s(1-e^{-\lambda t})}{s e^{-\lambda t}}$$

Value for $G_i(s,t)$ we get

$$G_i(s,t) = \left(\frac{1}{1+c_4}\right)^i$$

$$\boxed{G_i(s,t) = \left(\frac{s e^{-\lambda t}}{1 - s(1-e^{-\lambda t})}\right)^i}$$

$$G_i(s,t) = \left(\frac{s e^{-\lambda t}}{1 - s(1-e^{-\lambda t})} \right)^i$$

value of $p_{ij}(t)$ we have

$$G_i(s,t) = \sum_{j=i}^{\infty} p_{ij}(t) s^j$$

Instead of obtaining $p_{ij}(t)$ as $\frac{1}{j!} \partial_s G_i^{(j)}(s,t) |_{s=0}$, one can expand G_i directly and prove by induction that

$$\boxed{p_{ij}(t) = e^{-i\lambda t} \binom{j-1}{i-1} (1-e^{-\lambda t})^{j-i}} \quad (5)$$

That is assume (5) holds true for i.

Then

$$G_{i+1}(s,t) = G_i(s,t) \frac{s e^{-\lambda t}}{1 - s(1-e^{-\lambda t})}$$

$$= \left(\sum_{j=i}^{\infty} p_{ij}(t) s^j \right) s e^{-\lambda t} \sum_{k=0}^{\infty} (1-e^{-\lambda t})^k s^k$$

$$= e^{-i\lambda t} \sum_{j=i}^{\infty} \binom{j-1}{i-1} (1-e^{-\lambda t})^{j-i} s^j$$

$$\times \quad s e^{-\lambda t} \sum_{k=0}^{\infty} (1-e^{-\lambda t})^k s^k$$

Induction procedure Summarizing, we have

$$G_{i+1}(s,t) = e^{-i\lambda t}\sum_{j=i}^{\infty} \binom{j-1}{i-1}\left(1-e^{-\lambda t}\right)^{j-i} s^{j}$$

cv: $j=i+l$

$$\times \quad s\, e^{-\lambda t}\sum_{k=0}^{\infty} \left(1-e^{-\lambda t}\right)^{k} s^{k}$$

$$= e^{-(i+1)\lambda t}\, s^{i+1} \sum_{l=0}^{\infty} \overbrace{\binom{i-1+l}{i-1}\left(1-e^{-\lambda t}\right)^{l}}^{\equiv a_l} s^{l}$$

$$\times \quad \sum_{k=0}^{\infty} \overbrace{\left(1-e^{-\lambda t}\right)^{k}}^{\equiv b_k} s^{k}$$

$$= e^{-(i+1)\lambda t}\, s^{i+1} \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} a_n\, b_{m-n}\right) s^{m}$$

Computing $\sum_{n=0}^{m} a_n\, b_{m-n}$ we obtain

$$G_{i+1}(s,t) = \sum_{j=i+1}^{\infty} p_{i+1,j}(t)\, s^{j},$$

with

$$p_{i+1,j} = e^{-(i+1)t}\binom{j-1}{i}\left(1-e^{-\lambda t}\right)^{j-i-1}$$

Expected value Recall the value

$$G_i(s,t) = \left(\frac{e^{-\lambda t} s}{-(1-e^{-\lambda t})s+1} \right)^i$$

Then

$\partial_s G_i(s,t)$

$$= i \left(\frac{s\, e^{-\lambda t}}{1-s(1-e^{-\lambda t})} \right)^{i-1} \frac{e^{-\lambda t}}{(1-s(1-e^{-\lambda t}))^2}$$

$$= i \frac{e^{-\lambda i t}}{(1-s(1-e^{-\lambda t}))^{i-1}} s^{i-1}$$

Hence

$$\boxed{\mathbb{E}[B(t) \mid B(0)=i]}$$

$$= \partial_s G_i(s,t)\big|_{s=1}$$

$$= i \frac{e^{-\lambda i t}}{e^{-\lambda(i-1)t}} \quad \boxed{= i\, e^{\lambda t}}$$