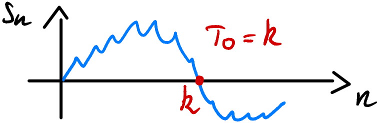
Notation (1)



Return time to 0: We set $T_0 = \infty$ if there is no return to 0, and

$$T_0 = \inf\{n > 0;\ S_n = 0\}$$

Probability to be at origin after n steps: We set

$$p_0(n) = \mathbf{P}(S_n = 0)$$

Probability that 1st return occurs after n steps: Define

$$f_0(n) = \mathbf{P}(T_0 = n) = \mathbf{P}(S_1 \neq 0, \ldots, S_{n-1} \neq 0, S_n = 0)$$

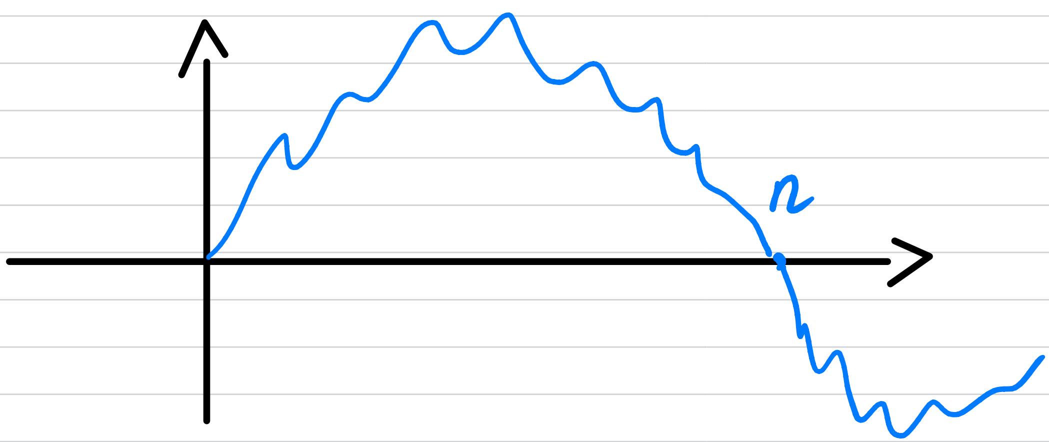
<u>Rmk</u> It is easier to compute

$$p_0(n) = P(S_n = 0) \longrightarrow$$ involves S at time n only

than

$$f_0(n) \equiv P(T_0 = n) = P(S_1 \neq 0, S_2 \neq 0, \ldots, S_{n-1} \neq 0, S_n = 0)$$

involves the whole past of (S_k) up to time n



Notation (2)

$p_0(n) = P(X_n = 0)$
$f_0(n) = P(T_0 = n)$

Generating functions: We set

generating function for the pmf of T_0

$$P_0(s) = \sum_{n=0}^{\infty} p_0(n) s^n \,, \qquad F_0(s) = \sum_{n=1}^{\infty} f_0(n) s^n$$

Probabilistic interpretation: We have

potentially $P(T_0 = \infty) > 0$

$$F_0(s) = \mathbf{E}\left[s^{T_0} \right]$$

Warning: T_0 is a defective random variable. Thus we have

- $s^{T_0} = 0$ if $T_0 = \infty$ if $s \in [0, 1)$
- This is also valid as $s \nearrow 1$ (hence "$1^\infty = 0$" here)
- Thus $F_0(1) = \mathbf{P}(T_0 < \infty)$

$\lim_{s \nearrow 1} F_0(s) = 1 - P(T_0 = \infty)$
$= P(T_0 < \infty)$

Justification for $F_0(1) = P(T_0 < \infty)$

$$F_0(s) = \sum_{n=1}^{\infty} f_0(n)\, s^n$$

Thus by Abel's thm

$$F_0(1) = \lim_{s \nearrow 1} F_0(s) = \lim_{s \nearrow 1} \sum_{n=1}^{\infty} f_0(n)\, s^n$$

$$= \sum_{n=1}^{\infty} f_0(n) = \sum_{n=1}^{\infty} P(T_0 = n)$$

$$= P(T_0 \in \{1, 2, 3, \ldots\})$$

$$\boxed{F_0(1) = P(T_0 < \infty)}$$

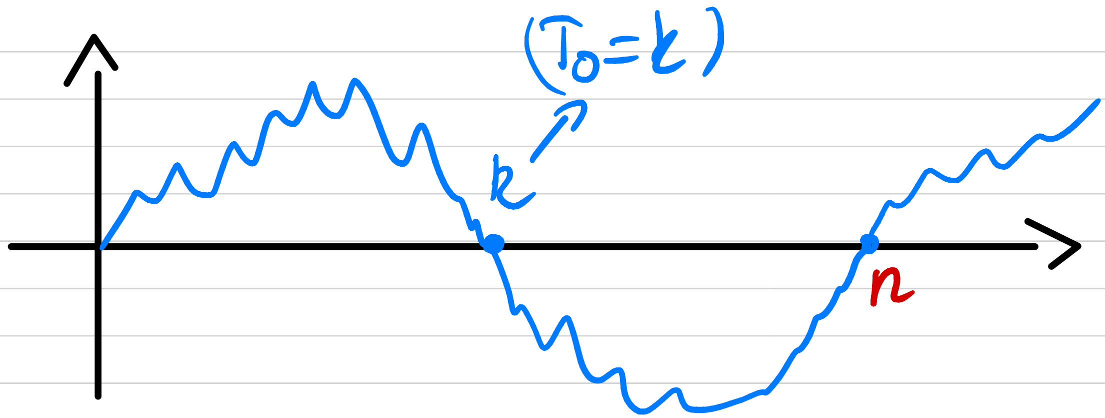
T_0 is a defective random variable : means that $T_0 \in \{1, 2, \dots, \} \cup \{\infty\}$

Computing P_0 and F_0

Theorem 11.

Let S_n be the random walk with parameters p and $q = 1 - p$. Then

1. P_0 and F_0 satisfy
$$P_0(s) = 1 + P_0(s)F_0(s)$$
2. P_0 verifies
$$P_0(s) = \frac{1}{(1 - 4pqs^2)^{1/2}}$$
3. F_0 is given by
$$F_0(s) = 1 - \left(1 - 4pqs^2\right)^{1/2}$$



Set $A = (S_n = 0)$

We have $\;P(S_n = 0) = P\Big((S_n = 0) \cap \overbrace{\bigcup_{k=1}^{n} (T_0 = k)}^{(T_0 \leq n)} \Big)$

If $(S_n = 0) \Rightarrow (T_0 \leq n)$

Thus $(S_n = 0) \subset (T_0 \leq n)$

$$\Rightarrow P(S_n = 0) = P\big((S_n = 0) \cap (T_0 \leq n)\big)$$

$$A \subset B \Rightarrow P(A) = P(A \cap B)$$

$P(A \cap B) = P(A|B)\,P(B)$

disjoint sets

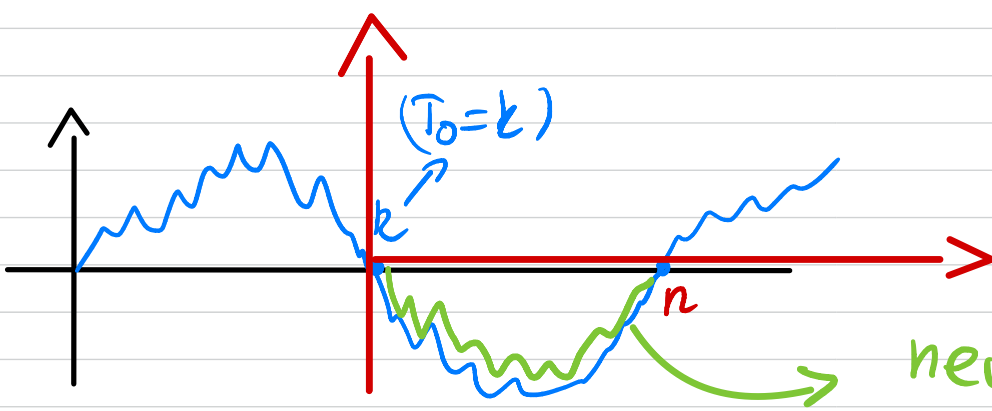
$$\overbrace{P(S_n = 0)}^{p_0(n)} = P\left((S_n=0) \cap \overbrace{\bigcup_{k=1}^{n} (T_0 = k)} \right)$$

$$= \sum_{k=1}^{n} P\big((S_n = 0) \cap (T_0 = k) \big)$$

$$= \sum_{k=1}^{n} \underbrace{P(S_n = 0 \mid T_0 = k)}_{p_0(n-k)} \; \underbrace{P(T_0 = k)}_{f_0(k)}$$

<u>We have obtained:</u>

$$\underbrace{p_0(n) = \sum_{k=1}^{n} p_0(n-k)\, f_0(k)}_{\text{almost a convolution } (k=0 \text{ is missing})}$$



new random walk starting from 0

Summary

$$p_0(n) = \sum_{k=1}^{n} p_0(n-k) f_0(k) \qquad + \quad p_0(0) = 1$$

Taking

$$\sum p_0(n) s^n = \sum \left(\sum_{k=1}^{n} p_0(n-k) f_0(k) \right) s^n$$

+ same computations as for the convolution

we get

$$\boxed{P_0(s) = 1 + P_0(s) F_0(s)}$$

Rmk The steps for the rw are X_i
X_i is not exactly a $\mathcal{B}(p)$ r.v.
However, one can write

$$X_i = 2Y_i - 1 \quad , \text{ with } \quad Y_i \sim \mathcal{B}(p)$$

Indeed, if $Y_i = 0$, then $X_i = -1$ (prob q)
if $Y_i = 1$, then $X_i = 1$ (prob p)

Compute $p_0(n)$.

$$p_0(n) = P(S_n = 0) = P\left(\sum_{k=1}^n X_k = 0\right)$$

$$= P\left(\sum_{k=1}^n (2Y_k - 1) = 0\right) = P\left(2 \underbrace{\sum_{k=1}^n Y_k}_{\mathrm{Bin}(n,p)} - n = 0\right)$$

$$= P\left(\sum_{k=1}^n Y_k = \frac{n}{2}\right) = \begin{cases} \binom{n}{n/2} p^{n/2} q^{n/2} & \text{if } n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Proof of Theorem 11 (1)

Events: We set

$$A = (S_n = 0), \qquad B_k = (T_0 = k)$$

Decomposition for A: We have

$$A = A \cap \left(\bigcup_{k=1}^{n} B_k \right) = \bigcup_{k=1}^{n} (A \cap B_k)$$

Decomposition for $\mathbf{P}(A)$: We get

$$\mathbf{P}(A) = \sum_{k=1}^{n} \mathbf{P}\left(A \cap B_k\right) = \sum_{k=1}^{n} \mathbf{P}\left(A \middle| B_k\right) \mathbf{P}(B_k) \tag{1}$$

Proof of Theorem 11 (2)

Convolution relation: Equation (1) can be read as

$$p_0(n) = \sum_{k=1}^{n} p_0(n-k) f_0(k), \quad \text{for} \quad n \geq 1, \quad \text{and} \quad p_0(0) = 1$$

Expression with generating functions: We get

$$P_0(s) = 1 + P_0(s) F_0(s)$$

Proof of Theorem 11 (3)

Computing $p_0(n)$: We have

1. For n odd,
$$p_0(n) = 0$$
2. For n even, one argue that
   - $(S_n = 0) \iff$ equal # steps up and steps down
   - There is $\binom{n}{n/2}$ ways to choose the up steps
   - Probability of each sequence leading to 0: $(p\,q)^{n/2}$

Thus for n even we have

$$p_0(n) = \mathbf{P}(S_n = 0) = \binom{n}{n/2}(p\,q)^{n/2}$$