Aim Get an expression for

$f_0(n) = P(T_0 = n)$ $\quad P_0(n) = P(S_n = 0)$

We have proved

(i) $P_0(s) = 1 + P_0(s) F_0(s)$

(ii) $p_0(n) = \begin{cases} \binom{n}{n/2} (pq)^{n/2} & \text{if } n \text{ even} \\ 0 & \text{otherwise} \end{cases}$

Strategy (i) Compute $P_0(s)$

(ii) From there, compute $F_0(s)$

$$p_0(n) = \begin{cases} \binom{n}{n/2} (pq)^{n/2} & \text{if } n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$P_0(s) = \sum_{n=0}^{\infty} p_0(n)\, s^n$$

$$= \sum_{m=0}^{\infty} \binom{2m}{m} (pq)^m \; s^{2m}$$

$$\boxed{P_0(s) = \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2} (pqs^2)^m}$$

$$P_0(s) = \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2} (pqs^2)^m \qquad (1)$$

Taylor identity

$$\frac{1}{(1+x)^{\frac{1}{2}}} = \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2} \left(-\frac{x}{4}\right)^m$$

Identifying with (1) :

$$-\frac{x}{4} = pqs^2 \iff x = -4pqs^2$$

$$\boxed{P_0(s) = \frac{1}{(1-4pqs^2)^{\frac{1}{2}}}}$$

$$P_0(s) = \frac{1}{(1-4pqs^2)^{\frac{1}{2}}}$$

Computation of $F_0(s)$: we have seen

$$P_0(s) = 1 + P_0(s)\, F_0(s)$$

$$\Longleftrightarrow \quad F_0(s) = \frac{P_0(s) - 1}{P_0(s)} = \frac{\frac{1}{(1-4pqs^2)^{\frac{1}{2}}} - 1}{\frac{1}{(1-4pqs^2)^{\frac{1}{2}}}}$$

$$\boxed{F_0(s) = 1 - (1-4pqs^2)^{\frac{1}{2}}}$$

Rmk From $F_0(s)$, we could derive all the values

$f_0(n) = P(T_0 = n)$, by differentiating $F_0(s)$

Proof of Theorem 11 (4)

First expression for P_0: We have found

$$P_0(n) = \sum_{m=0}^{\infty} \binom{2m}{m} (p\,q)^m s^{2m} = \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2} (p\,q\,s^2)^m$$

A binomial series: We have

$$\frac{1}{(1+x)^{1/2}} = \sum_{m=0}^{\infty} \frac{(-1)^m (2m)!}{4^m (m!)^2} x^m = \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2} \left(-\frac{x}{4}\right)^m$$

Second expression for P_0: We get

$$P_0(s) = \frac{1}{(1-4pqs^2)^{1/2}}$$

Proof of Theorem 11 (5)

Summary: We have obtained

$$\begin{aligned} P_0(s) &= 1 + P_0(s)F_0(s) \\ P_0(s) &= \frac{1}{(1-4pqs^2)^{1/2}} \end{aligned}$$

Conclusion: We easily get

$$F_0(s) = 1 - \left(1 - 4pqs^2\right)^{1/2}$$

$F_0(s) = 1 - (1-4pqs^2)^{\frac{1}{2}}$ g.f of T_0

Expression for $P(T_0 < \infty)$ in terms of F_0

We have seen, for defective r.v, that

$$P(T_0 < \infty) = F_0(1)$$

$$= 1 - (1-4pq)^{\frac{1}{2}}$$

Thus

$$P(T_0 < \infty) = 1 - (1 - 4p(1-p))^{\frac{1}{2}}$$

$$= 1 - (1 - 4p + 4p^2)^{\frac{1}{2}}$$

$$= 1 - ((2p-1)^2)^{\frac{1}{2}}$$

$\sqrt{a^2} = |a|$

$$= 1 - |2p-1|$$

$$\boxed{P(T_0 < \infty) = 1 - |p-q|}$$

$P(T_0 < \infty) = 1 - |p-q| < 1$ if $p \neq q$

Rmk $T_0 \geq 0$, and $p \neq q$ we have

$$P(T_0 = \infty) > 0 \quad \text{if } p \neq q$$

$\Rightarrow$ $\boxed{E[T_0] = \infty}$

$\Rightarrow$ The interesting case for $E[T_0]$ is when $p = q = \frac{1}{2}$

Case $p = \frac{1}{2}$. We have

$$F_0(s) = 1 - (1 - 4pqs^2)^{\frac{1}{2}} \quad \& \quad pq = \frac{1}{4}$$

$$F_0(s) = 1 - (1 - s^2)^{\frac{1}{2}}$$

Computation of $E[T_0]$

- $F_0(s) = 1 - (1-s^2)^{\frac{1}{2}}$
- $E[T_0] = F_0'(1)$

We have $F_0'(s) = \frac{1}{2} \dfrac{2s}{(1-s^2)^{\frac{1}{2}}}$

$$F_0'(s) = \frac{s}{(1-s^2)^{\frac{1}{2}}}$$

Thus $F_0'(1) = \lim_{s \nearrow 1} F_0'(s) = \infty$

We have found $\boxed{E[T_0] = \infty}$

Interpretation for $p > q$ $\left(p > \frac{1}{2}\right)$

$$\frac{S_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad X_i \text{ iid r.v}$$

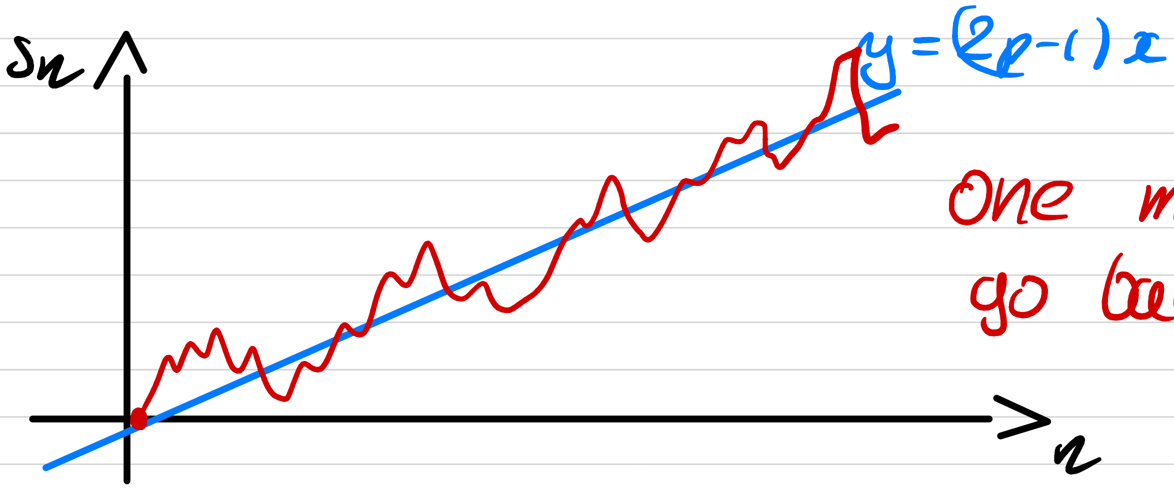
Law of large numbers : $\frac{S_n}{n} \longrightarrow E[X_1]$

and $E[X_1] = p \times 1 + (1-p)(-1) = P(X_1 = 1) \times 1 + P(X_1 = -1) \times (-1)$

$$= 2p - 1 > 0$$

Thus

$$S_n \simeq (2p-1)\, n$$



One might never go back to 0

Interpretation $p = \frac{1}{2}$

Eventually we are sure to go back to 0, but it can take a very long time

Proof of Proposition 12 (1)

Expression for F_0: We have seen

$$F_0(s) = 1 - \left(1 - 4pqs^2\right)^{1/2}$$

Expression for $\mathbf{P}(T_0 < \infty)$: We have also seen that

$$\mathbf{P}(T_0 < \infty) = F_0(1)$$

Hence

$$\begin{aligned}\mathbf{P}(T_0 < \infty) &= F_0(1) \\ &= 1 - (1 - 4pq)^{1/2} \\ &= 1 - |2p - 1| \\ &= 1 - |p - q|\end{aligned}$$

Proof of Proposition 12 (2)

F_0 for $p = 1/2$: When $p = q = \frac{1}{2}$ we have

$$F_0(s) = 1 - \left(1 - s^2\right)^{1/2}$$

Expression for $\mathbf{E}[T_0]$: We have seen that

$$\mathbf{E}[T_0] = F_0'(1)$$

Computation of F_0': We get

$$F_0'(s) = \frac{s}{\left(1 - s^2\right)^{1/2}}$$

Conclusion: We have

$$\mathbf{E}[T_0] = F_0'(1) = \infty$$