Vocabulary

If $N(0) = \#$ visits to 0. In the recurrent case $P(N(0) = \infty) = 1$

Rmk: In that case, # visits to 0 is $= \infty$

Persistent or recurrent: the random walk is said to be recurrent
$\hookrightarrow$ iff $\mathbf{P}(T_0 < \infty) = 1$

Transient: the random walk is said to be transient
$\hookrightarrow$ iff $\mathbf{P}(T_0 < \infty) < 1$ Rmk: In that case # visits to 0 is always finite

Summarizing our result: We have seen that $P(N(0) = \infty) = 0$

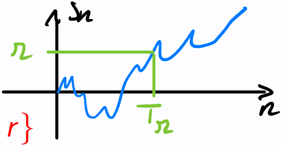
$$\text{Random walk is persistent} \iff p = \frac{1}{2}$$

Visits to point r

$r > 0$

Definition 13.

The first time to visit r is defined by



$$T_r = \inf\{n > 0;\ S_n = r\}$$

Then we set

$$f_r(n) = \mathbf{P}(T_r = n) = \mathbf{P}(S_1 \neq r, \ldots, S_{n-1} \neq r, S_n = r)$$

and

$$F_r(s) = \sum_{n=1}^{\infty} f_r(n) s^n$$

Generating function for T_r

Theorem 14.

For $r \geq 1$ we have

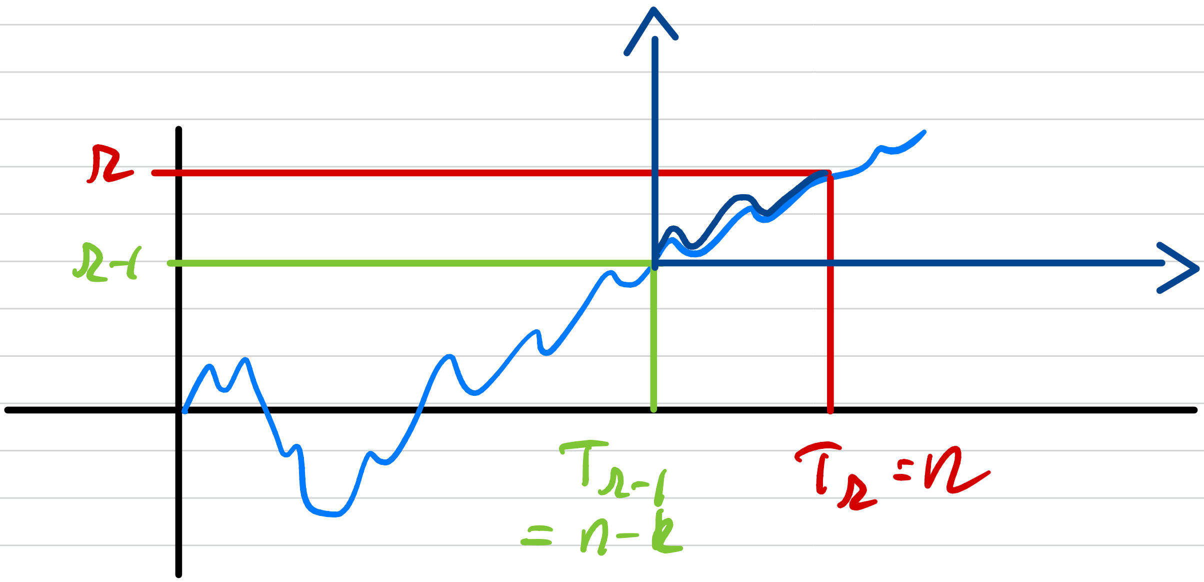
$$F_r(s) = [F_1(s)]^r$$

with

$$F_1(s) = \frac{1 - (1 - 4pqs^2)^{1/2}}{2q}$$

Strategy : . Decompose $A = (T_r = n)$

. Before reaching r, we need to reach $r-1$



$r > 1 \quad (\Rightarrow r-1 \geq 1)$

Decomposition:

$$P(T_r = n) = P\Big((T_r = n) \cap \overbrace{\bigcup_{k=1}^{n-1} (T_{r-1} = n-k)}^{\text{disjoints}}\Big)$$

$$= \sum_{k=1}^{n-1} P\big((T_r = n) \cap (T_{r-1} = n-k)\big)$$

$$= \sum_{k=1}^{n-1} \underbrace{P(T_r = n \mid T_{r-1} = n-k)}_{f_1(k)} \overbrace{P(T_{r-1} = n-k)}^{f_{r-1}(n-k)}$$

Summary: We have found $r > 1 \quad n \geq 1$

$$f_r(n) = \sum_{k=1}^{n-1} f_1(k)\, f_{r-1}(n-k)$$

Initial condition $f_r(0) = P(T_r = 0) = 0$

By convolution,

$$F_r(s) = F_1(s)\, F_{r-1}(s)$$

Recursion we find

$$\boxed{F_r(s) = [F_1(s)]^r}$$

Conditioning to compute F_1

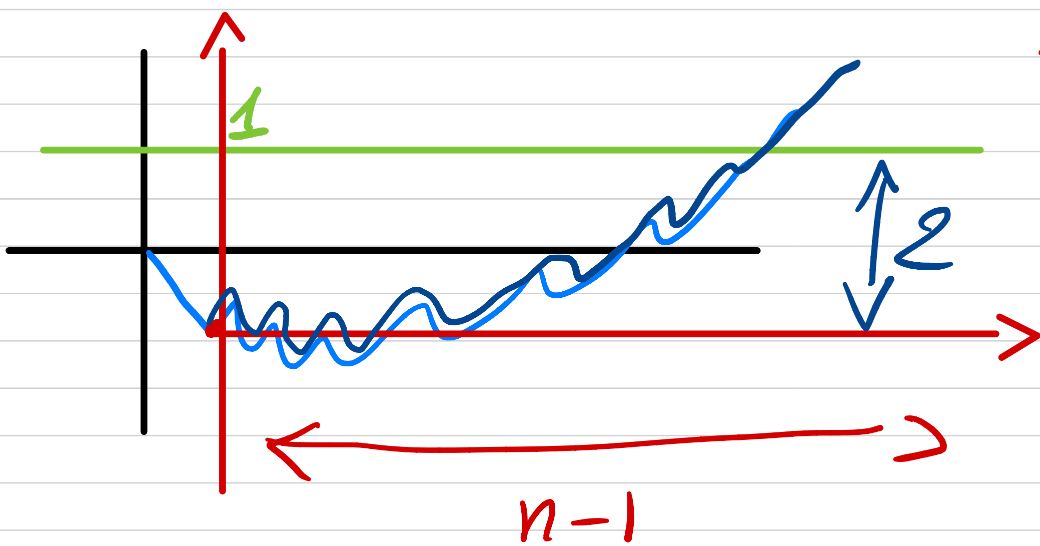
Rmk: $\overset{=X_1}{S_1} \in \{-1, 1\}$

Hyp: $n > 1$

$$f_1(n) = P(T_1 = n)$$

$$= P(T_1 = n \mid S_1 = 1)\, P(S_1 = 1) + P(T_1 = n \mid S_1 = -1) \times P(S_1 = -1)$$

$$= 0 \times p + \underbrace{P(T_1 = n \mid S_1 = -1)}_{f_2(n-1)}\, q$$



$$\boxed{f_1(n) = f_2(n-1)\, q}$$

we have obtained, for $n > 1$ first step is up

$$f_1(n) = q\, f_2(n-1) \quad \& \quad f_1(1) = P(T_1 = 1) = p$$

Sum over n

$$F_1(s) = \sum_{n=0}^{\infty} f_1(n)\, s^n = ps + \sum_{n=2}^{\infty} \overbrace{q\, f_2(n-1)}^{f_1(n)}\, s^n \qquad s^{n-1}\, s$$

$$= ps + qs \sum_{n=2}^{\infty} f_2(n-1)\, s^{n-1} \qquad \text{CV: } n := n-1$$

$$= ps + qs \sum_{n=0}^{\infty} f_2(n)\, s^n$$

we get

$$F_1(s) = ps + qs\, F_2(s) \qquad F_2(s) = (F_1(s))^2$$

$$\boxed{F_1(s) = ps + qs\, (F_1(s))^2} \rightarrow \text{quadratic eq.}$$

Proof of Theorem 14 (1)

Events: We set, for $r > 1$,

$$A = (T_r = n), \qquad B_k = (T_{r-1} = n - k)$$

Decomposition for A: We have

$$A = A \cap \left(\bigcup_{k=1}^{n-1} B_k \right) = \bigcup_{k=1}^{n-1} (A \cap B_k)$$

Decomposition for $\mathbf{P}(A)$: We get

$$\mathbf{P}(A) = \sum_{k=1}^{n-1} \mathbf{P}\left(A \cap B_k\right) = \sum_{k=1}^{n-1} \mathbf{P}\left(A \middle| B_k\right) \mathbf{P}(B_k) \tag{2}$$

Proof of Theorem 14 (2)

Convolution relation: Equation (2) can be read as

$$f_r(n) = \sum_{k=1}^{n-1} f_1(k) f_{r-1}(n-k), \quad \text{for} \quad n \geq 1, \quad \text{and} \quad f_r(0) = 0$$

Expression with generating functions: We get

$$F_r(s) = F_1(s) F_{r-1}(s)$$

Conclusion for F_r: Iterating the above relation we get

$$F_r(s) = [F_1(s)]^r$$

Proof of Theorem 14 (3)

Conditioning on X_1: For $n > 1$ we have

$$\begin{aligned} \mathbf{P}(T_1 = n) &= \mathbf{P}(T_1 = n | X_1 = 1)p + \mathbf{P}(T_1 = n | X_1 = -1)q \\ &= 0 + \mathbf{P}\left(\text{1st visit to 1 takes } n-1 \text{ steps} | S_0 = -1\right) q \\ &= \mathbf{P}(T_2 = n-1)q \end{aligned}$$

Relation on pmf's: We get, for $n > 1$

$$f_1(n) = q f_2(n)$$

Relation on generating functions: Multiplying by s^n we obtain

$$\begin{aligned} F_1(s) &= ps + sqF_2(s) \\ &= ps + sq\left(F_1(s)\right)^2 \end{aligned}$$

Proof of Theorem 14 (4)

Recall: We have obtained

$$F_1(s) = ps + sq\left(F_1(s)\right)^2$$

Expression for F_1:
Solving for $F_1(s)$ in the quadratic equation we get

$$F_1(s) = \frac{1 - \left(1 - 4pqs^2\right)^{1/2}}{2q}$$

Visits to the upper half plane

Proposition 15.

Let S_n be the random walk
$\hookrightarrow$ with parameters p and $q = 1 - p$.

Then

if $p>q$

$$\mathbf{P}(\text{At least one visit to the upper half plane}) = \min\left(1, \frac{p}{q}\right)$$

if $p<q$