Summary

$r \geq 1$

$$T_r = \inf\{n > 0;\ S_n = r\}$$

Then

(i) $F_r(s) = (F_1(s))^r$

(ii) $$\boxed{F_1(s) = \frac{1 - (1 - 4pqs^2)^{\frac{1}{2}}}{2q}}$$

Visits to the upper half plane

Proposition 15.

Let S_n be the random walk
$\hookrightarrow$ with parameters p and $q = 1 - p$.

Then

$$\mathbf{P}(\text{At least one visit to the upper half plane}) = \min\left(1, \frac{p}{q}\right)$$

$$= \begin{cases} 1 & \text{if } p \geq \frac{1}{2} \\ \frac{p}{q} & \text{if } p < \frac{1}{2} \end{cases}$$

Relation to T_1

Let

$A =$ (At least one visit to upper half plane)

$=$ (we hit level 1)

$= (T_1 < \infty)$

Relation to F_1

we have seen

$$P(T_1 < \infty) = F_1(1) = \frac{1-(1-4pq)^{\frac{1}{2}}}{2q}$$

$$= \frac{1-(1-4p(1-p))^{\frac{1}{2}}}{2q}$$

$$= \frac{1-[(2p-1)^2]^{\frac{1}{2}}}{2q} = \frac{1-|2p-1|}{2q}$$

Computation of $P(A)$

$$P(A) = P(T_1 < \infty) = \frac{1 - |2p-1|}{2(1-p)}$$

$\Leftrightarrow p > q$

Case $p \geq \frac{1}{2}$: we get

$$\boxed{P(A) =} \quad \frac{1 - 2p + 1}{2(1-p)} = \frac{2 - 2p}{2 - 2p} \quad \boxed{= 1}$$

Case $p < \frac{1}{2}$: we get

$$\boxed{P(A) =} \quad \frac{1 - 1 + 2p}{2q} = \frac{2p}{2q} \quad \boxed{= \frac{p}{q}}$$

Proof of Proposition 16

Notation: Set

$$A = \text{At least one visit to the upper half plane}$$

Expression with generating function: We have

$$\begin{aligned}
\mathbf{P}(A) &= \mathbf{P}(T_1 < \infty) \\
&= F_1(1) \\
&= \frac{1 - |p - q|}{2q}
\end{aligned}$$

Conclusion: Separating cases $p > q$ and $p \le q$ we get

$$\mathbf{P}(A) = \min\left(1, \frac{p}{q}\right)$$

Hitting time theorem

Theorem 16.

Let

- S_n be the random walk with parameters p and $q = 1 - p$
- $b \in \mathbb{Z}^*$ and $n \geq 1$
- $T_b = \inf\{n > 0;\ S_n = b\}$

Then

$$\overbrace{\mathbf{P}(T_b = n)}^{\text{difficult to compute}} = \frac{|b|}{n} \overbrace{\mathbf{P}(S_b = n)}^{\text{easy to compute}}$$

Outline

Calendar has changed

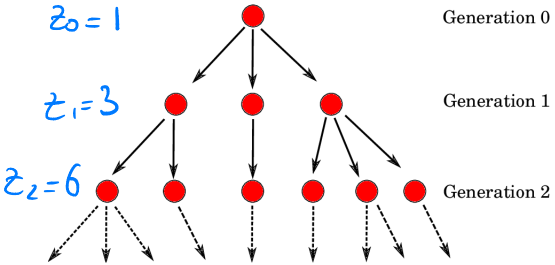
(i) HW schedule

(ii) Midterm : 3/6

Model

Model for population evolution:

- $Z_n \equiv \#$ individuals of n-th generation
- At n-th generation: each member gives birth
  $\hookrightarrow$ To a $\#$ individuals of $(n+1)$-th generation
- Family size: random variable



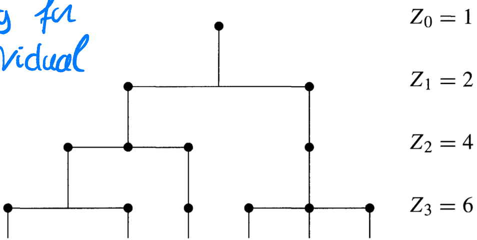
Assumptions on the model

all equally distributed

Main hypotheses:

1. Family sizes form collection of $\perp\!\!\!\perp$ random variables
2. Family sizes have same pmf f
   $\hookrightarrow$ with generating function G
3. $Z_0 = 1$

\# offspring for each individual



$Z_2 = \sum_{i=1}^{Z_1} Y_i$

\# offsprings

Generating functions for random sums

Theorem 17.

Let

- $\{X_j;\, j \geq 1\}$ sequence of i.i.d random variables
- $G_X \equiv$ common generating function $= G_{X_1}$
- N random variable, with $N \perp\!\!\!\perp (X_j)_{j\geq 1}$ and $N \in \{0, 1, \ldots\}$
- $G_N \equiv$ generating function for N
- $Z = \sum_{j=1}^{N} X_j$

Then

$$G_Z(s) = G_N(G_X(s))$$

Notation: If A event, $\mathbb{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$

$$Z = \sum_{i=1}^{N} X_i$$

Generating function $\quad G_Z(s) = E[s^Z]$

$$\boxed{G_Z(s)} = E\left[s^{\sum_{i=1}^{N} X_i}\right]$$

$$\sum_{n=0}^{\infty} \mathbb{1}_{N=n} = 1$$

$$= E\left[s^{\sum_{i=1}^{N} X_i} \sum_{n=0}^{\infty} \mathbb{1}_{(N=n)}\right]$$

$$= \sum_{n=0}^{\infty} E\left[s^{\sum_{i=1}^{N} X_i} \mathbb{1}_{(N=n)}\right]$$

$$= \sum_{n=0}^{\infty} E\left[s^{\sum_{i=1}^{n} X_i} \mathbb{1}_{(N=n)}\right] \quad \text{(since } (X_i) \perp\!\!\!\perp N \text{)}$$

$$= \sum_{n=0}^{\infty} E\left[s^{\sum_{i=1}^{n} X_i}\right] \underbrace{E[\mathbb{1}_{N=n}]}_{P(N=n)} \quad \text{all } \perp\!\!\!\perp$$

$$= \sum_{n=0}^{\infty} \prod_{i=1}^{n} \underbrace{E[s^{X_i}]}_{=G_X(s)} \; f_N(n) \quad \text{pmf of } N$$

$$= \sum_{n=0}^{\infty} (G_X(s))^n f_N(n) \; \boxed{= G_N(G_X(s))}$$

Proof of Theorem 17

Computation: We have

$$
\begin{aligned}
G_Z(s) &= \mathbf{E}[s^Z] \\
&= \sum_{n=0}^{\infty} \mathbf{E}\left[s^Z \,|\, N=n\right] \mathbf{P}(N=n) \\
&= \sum_{n=0}^{\infty} \mathbf{E}\left[s^{\sum_{j=1}^{N} X_j} \,|\, N=n\right] \mathbf{P}(N=n) \\
&= \sum_{n=0}^{\infty} \mathbf{E}\left[s^{\sum_{j=1}^{n} X_j}\right] \mathbf{P}(N=n) \\
&= \sum_{n=0}^{\infty} \left(G_X(s)\right)^n f_N(n) \\
&= G_N(G_X(s))
\end{aligned}
$$

Generating function for the branching process

Theorem 18.

For the branching process, recall that

- $Z_n = \#$ individuals of n-th generation
- $G =$ generating function for the offspring $f = G_{X_1}$

We set

$$G_n(s) = \mathbf{E}\left[s^{Z_n}\right] = \text{gen. fct for } Z_n$$

Then

$$G_{m+n}(s) = G_m(G_n(s)) = G_n(G_m(s))$$

Thus

$$G_n(s) = G^{\circ(n)}(s) = \underbrace{G(G(\cdots G(s))\cdots)}_{n \text{ times}}$$