Generating function for the branching process

Theorem 18.

For the branching process, recall that

- $Z_n = \#$ individuals of n-th generation
- $G =$ generating function for the offspring f

We set

$$G_n(s) = \mathbf{E}\left[s^{Z_n}\right]$$

Then

$$G_{m+n}(s) = G_m(G_n(s)) = G_n(G_m(s))$$

Thus

$$G_n(s) = G^{\circ(n)}(s)$$

Recursion Recall that $Z_{n+1} = \sum_{j=1}^{Z_n} Y_i^{(n+1)}$

And Z_n has pgf G_n

$\{Y_i^{(n+1)}, i \geq 1\}$ have pgf G

Aplying Thm 17, we get

$$\boxed{G_{Z_{n+1}}(s) = G_n(G(s))}$$

$P(Z_0 = 1) = 1$

$P(Z_0 = k) = 0$ if $k \neq 1$

Initial condition For $n = 0$, $Z_0 = 1$

$$G_0(s) = \sum_{k=0}^{\infty} P(Z_0 = k)\, s^k = 1 \times s^1 = s$$

$$\boxed{G_0(s) = s}$$

conclusion

$G_0(s) = s$, $G_1(s) = G(G_0(s)) = G(s) = G^{o(1)}(s)$

... $G_n(s) = G^{o(n)}(s)$

Proof of Theorem 18 (1)

Decomposition of Z_{n+m}: Write

$$\begin{aligned} Z_{n+m} &= Y_1 + \cdots + Y_{Z_m} \\ &= \sum_{j=1}^{Z_m} Y_j, \end{aligned}$$

where

$Y_j = \#$ individuals in generation $(n+m)$ which stem from individual j in m-th generation

Proof of Theorem 18 (2)

Recall:

$$Z_{n+m} = \sum_{j=1}^{Z_m} Y_j$$

Information on the random variables Y_j:

- Y_j's are independent
- Y_j's are independent of Z_m
- $Y_j \overset{(d)}{=} Z_n$

Application of Theorem 17:

$$G_{m+n}(s) = G_m(G_{Y_1}(s)) = G_m(G_n(s))$$

Moments of Z_n

Intuition: $\mu = 1 \Rightarrow$ survival

Proposition 19.

For the branching process with offspring $Z_1 \sim f$ set

$$\mu = \mathbf{E}[Z_1], \qquad \sigma^2 = \mathbf{Var}(Z_1)$$

Then

$$\mathbf{E}[Z_n] = \mu^n$$

and

$$\mathbf{Var}(Z_n) = \begin{cases} n\,\sigma^2 & \text{if } \mu = 1 \\ \frac{\sigma^2(\mu^n - 1)\mu^{n-1}}{\mu - 1} & \text{if } \mu \neq 1 \end{cases}$$

Remark on $E[Z_n]$

$$E[Z_n] = \mu^n$$

Thus $\mu < 1 \Rightarrow \lim_{n\to\infty} E[Z_n] = 0$

(on average, extinction)

$\mu = 1 \Rightarrow \lim_{n\to\infty} E[Z_n] = 1$

(on average, survival)

$\mu > 1 \Rightarrow \lim_{n\to\infty} E[Z_n] = \infty$

(on average, exponential growth)

Q: What does the variance of a r.v. measure? ↳ Fluctuations predictable / unpredictable

Remark on $Var(z_n)$

If $\mu = 1$, $Var(z_n) = n\sigma^2$

Thus $E[z_n] = 1$, Oscillations $(z_n) \simeq \sigma\sqrt{n}$

(z_n might get large, or $z_n \to 0$)

If $\mu < 1$

$$Var(z_n) = \frac{\sigma^2(\mu^n - 1)\mu^{n-1}}{\mu - 1} = \frac{1-\mu^n}{1-\mu}\sigma^2\mu^{n-1}$$

$$\sigma_{z_n} = (Var(z_n))^{1/2} = \left(\frac{1-\mu^n}{1-\mu}\right)^{\frac{1}{2}} \sigma \mu^{\frac{n-1}{2}} \xrightarrow{n \to 0} 0$$

$\to \frac{1}{1-\mu}$

Thus
$$\left.\begin{aligned} E[z_n] &\to 0 \\ \sigma_{z_n} &\to 0 \end{aligned}\right\} \Rightarrow \text{extinction}$$

Case $\mu > 1$

$$\mathrm{Var}(Z_n) = \frac{\sigma^2 (\mu^n - 1)\mu^{n-1}}{\mu - 1}$$

$$\sigma_{Z_n} = (\mathrm{Var}(Z_n))^{\frac{1}{2}} \simeq \frac{\sigma}{\sqrt{\mu-1}} \frac{1}{\sqrt{\mu}} \mu^n$$

Thus

$$\left.\begin{aligned} E[Z_n] &= \mu^n \\ \sigma_{Z_n} &\simeq C_{\mu,\sigma}\, \mu^n \end{aligned}\right)$$

Population explosion?

Proof of $E[Z_n] = \mu^n$

We know that $G_{n+1}(s) = G(G_n(s))$

we use this relation to get a recursion

on $E[Z_n] = G_n'(1)$

$$(G_{n+1}(s))' = (G(G_n(s)))'$$

$$G_{n+1}'(s) = G'(G_n(s))\, G_n'(s)$$

At $s=1$, we get $(G_n(1) = 1$ since Z_n finite)

$$E[Z_{n+1}] = G_{n+1}'(1) = G'(1)\, G_n'(1)$$

$$= \mu \, E[Z_n]$$

$$\Rightarrow \boxed{E[Z_{n+1}] = \mu \, E[Z_n]}$$

Recursion $\quad Z_0 = 1$

$$E[Z_0] = 1 \quad , \quad E[Z_{n+1}] = \mu E[Z_n]$$

$$\Rightarrow \quad \boxed{E[Z_n] = \mu^n}$$

Recursion for G''

$$\left(G'_{n+1}(s)\right)' = \left(G'(G_n(s)) \; G'_n(s)\right)'$$

$$\underbrace{G''_{n+1}(s) = G''(G_n(s))\left(G'_n(s)\right)^2 + G'(G_n(s)) \; G''_n(s)}$$

linear recursion for $u_n = G''_n(s)$

$$u_{n+1} = a_n + b_n u_n$$

we get

$$G_n''(1) = (\sigma^2 + \mu(\mu-1))\mu^{2(n-1)} + \mu G_{n-1}''(1)$$

$\Rightarrow$ expression for

$$G_n''(1) = E[Z_n(Z_n - 1)]$$

$\Rightarrow$ expression for

$$\boxed{Var(Z_n)}$$

Proof of Proposition 19 (1)

Method of computation: We use

$$\mathbf{E}[Z_n] = G_n'(1)$$

Recursive relation: Recall that

$$G_n(s) = G(G_{n-1}(s))$$

Proof of Proposition 19 (2)

Recall: We have

$$G_n(s) = G(G_{n-1}(s))$$

Differentiate: We have

$$G_n'(s) = G'\left(G_{n-1}(s)\right) G_{n-1}'(s)$$

Thus at $s = 1$ we get

$$\mathbf{E}[Z_n] = G'(1)\,\mathbf{E}[Z_{n-1}] = \mu\,\mathbf{E}[Z_{n-1}]$$

Conclusion: Since $\mathbf{E}[Z_0] = 1$, we get

$$\mathbf{E}[Z_n] = \mu^n$$

Proof of Proposition 19 (3)

Method for the variance: We use

$$\mathbf{E}[Z_n(Z_n - 1)] = G_n''(1)$$

or

$$\mathbf{Var}(Z_n) = G_n''(1) + G_n'(1) - \left(G_n'(1)\right)^2$$

Recursive relation: We differentiate twice the relation

$$G_n(s) = G(G_{n-1}(s))$$

We get a linear recursion (to be solved)

$$\begin{aligned} G_n''(1) &= G''(1)\left(G_{n-1}'(1)\right)^2 + G'(1)G_{n-1}''(1) \\ &= \left(\sigma^2 + \mu(\mu - 1)\right)\mu^{2(n-1)} + \mu G_{n-1}''(1) \end{aligned}$$