Continuous time Markov chains

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Elements of Stochastic Processes – MA 532

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Outline

1. Birth processes and the Poisson process
   - Poisson process
   - Birth processes
2. Continuous time Markov chain
   - General definitions and transitions
   - Generators
   - Classification of states

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Outline

A model for radioactive particles emission

Model for the process

- $N(t) \equiv \#$ particles emitted at time t
- $N = \{N(t);\, t \geq 0\}$
- $N(0) = 0$ and $N(t) \in \mathbb{N}$
- $N(s) \leq N(t)$ if $s \leq t$

Emission model:

- In $(t, t+h)$ there might/might not be emissions
- h small $\Longrightarrow$ likelihood of emission is $\simeq \lambda h$
  $\hookrightarrow$ with an intensity λ
- At most 1 emission if h is small

Definition of Poisson process

Definition 1.

Let

- $N = \{N(t);\ t \geq 0\}$ process with $N(0) = 0$ and $N(t) \in \mathbb{N}$

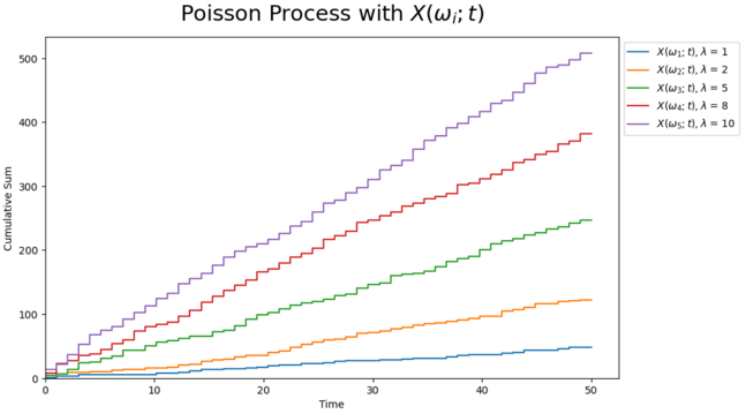
Then N is a Poisson process if

- $N(0) = 0$ and $t \mapsto N(t)$ is $\nearrow$
- Probability $\mathbf{P}\left(N(t+h) = n+m \middle| N(t) = n\right)$ of the form

$$\begin{cases} \lambda h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda h + o(h) & \text{if } m = 0 \end{cases}$$

- $N(t) - N(s) \perp\!\!\!\perp$ emissions on $[0, s]$

Paths of a Poisson process



Vocabulary

Terminology for Poisson processes:

- $N(t)$ is interpreted as a number of arrivals
- N is called counting process

Broader context:

- N is a simple example of continuous time Markov chain
- More general objects: in next section

Birth of Poisson process

3 independent discoveries:

- Lund, Sweden, 1903
  $\hookrightarrow$ Actuarial studies
- Erlang, Denmark, 1909
  $\hookrightarrow$ Telecommunication networks
- Rutherford, New Zealand, 1910
  $\hookrightarrow$ Particle emission

Marginal distribution

Theorem 2.

Let

- N Poisson process with intensity λ
- $t \geq 0$

Then

$$N(t) \sim \mathcal{P}(\lambda t),$$

that is for $j \in \mathbb{N}$ we have

$$\mathbf{P}(N(t) = j) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}$$

Proof of Theorem 2 (1)

Conditioning on a small interval: We have

$$
\begin{aligned}
&\mathbf{P}\left(N(t+h)=j\right) \\
&= \sum_{i\in S} \mathbf{P}\left(N(t+h)=j \,|\, N(t)=i\right) \mathbf{P}\left(N(t)=i\right) \\
&= \sum_{i\in S} \mathbf{P}\left((j-i) \text{ arrivals in } (t,t+h]\right) \mathbf{P}\left(N(t)=i\right) \\
&= \mathbf{P}\left(\text{no arrivals in } (t,t+h]\right)\, \mathbf{P}\left(N(t)=j\right) \\
&\quad + \mathbf{P}\left(\text{one arrival in } (t,t+h]\right)\, \mathbf{P}\left(N(t)=j-1\right) + o(h) \\
&= (1-\lambda h)\, \mathbf{P}\left(N(t)=j\right) + \lambda h\, \mathbf{P}\left(N(t)=j-1\right) + o(h)
\end{aligned}
$$

Proof of Theorem 2 (2)

Probability as a function: We set

$$p_j(t) = \mathbf{P}(N(t) = j)$$

Equation on small intervals: We have seen

$$\begin{aligned} p_0(t+h) &= (1 - \lambda h)\, p_0(t) + o(h) \\ p_j(t+h) &= \lambda h\, p_{j-1}(t) + (1 - \lambda h)\, p_j(t) + o(h) \end{aligned}$$

Equivalent form with differences:

$$\begin{aligned} p_0(t+h) - p_0(t) &= -\lambda h\, p_0(t) + o(h) \\ p_j(t+h) - p_j(t) &= \lambda h\, (p_{j-1}(t) - p_j(t)) + o(h) \end{aligned}$$

Proof of Theorem 2 (3)

Recall:

$$
\begin{aligned}
p_0(t+h) - p_0(t) &= -\lambda h\, p_0(t) + o(h) \\
p_j(t+h) - p_j(t) &= \lambda h\, (p_{j-1}(t) - p_j(t)) + o(h)
\end{aligned}
$$

Differentiating: We end up with a system of ode's

$$
\begin{aligned}
p_0'(t) &= -\lambda\, p_0(t) \\
p_j'(t+h) &= \lambda\, p_{j-1}(t) - \lambda\, p_j(t)
\end{aligned}
$$

Initial condition:

$$
p_j(0) = \delta_{j0} \equiv \mathbf{1}_{(j=0)}
$$

Proof of Theorem 2 (4)

Recall: We have obtained a system of ode's

$$
\begin{aligned}
p_0'(t) &= -\lambda\, p_0(t) \\
p_j'(t+h) &= \lambda\, p_{j-1}(t) - \lambda\, p_j(t)
\end{aligned}
$$

A family of generating functions: We set

$$G_t(s) = \mathbf{E}\left[s^{N(t)}\right] = \sum_{j=0}^{\infty} p_j(t) s^j$$

Strategy: From the system of ode's
$\hookrightarrow$ deduce a single ode for $t \mapsto G_t(s)$

Proof of Theorem 2 (5)

Differential equation for G: We have

$$
\begin{aligned}
\frac{\partial G_t(s)}{\partial t} &= \sum_{j=0}^{\infty} p_j'(t) s^j \\
&= -\lambda\, p_0(t) + \sum_{j=1}^{\infty} \left(\lambda\, p_{j-1}(t) - \lambda\, p_j(t)\right) s^j \\
&= -\lambda\, G_t(s) + \lambda s \sum_{j=1}^{\infty} p_{j-1}(t) s^{j-1} \\
&= -\lambda\, G_t(s) + \lambda s G_t(s) \\
&= \lambda(s-1) G_t(s)
\end{aligned}
$$

Proof of Theorem 2 (6)

Recall: $u_t \equiv G_t(s)$ verifies

$$u' = \lambda(s-1)u, \qquad u_0 = 1$$

Expression for $G_t(s)$: We find

$$G_t(s) = \exp\left(\lambda(s-1)t\right)$$

Conclusion:

$$N(t) \sim \mathcal{P}(\lambda t),$$

Relation with binomial random variables

Another way to prove $N(t) \sim \mathcal{P}(\lambda t)$:

1. Partition $[0, t]$ in subintervals $[(\ell - 1)h, \ell h]$
2. On each subinterval, set $Z_\ell = \mathbf{1}_{(\text{arrival in } [(\ell-1)h, \ell h])}$
3. We have that $\{Z_\ell;\ \ell \geq 1\}$ is i.i.d with common law $\mathcal{B}(\lambda h)$
4. We have $N(t) \simeq \sum_{\ell=1}^{t/h} Z_\ell$, thus

$$N(t) \simeq \text{Bin}\left(\frac{t}{h};\ \lambda h\right) \xrightarrow{h \to 0} \mathcal{P}(\lambda t)$$

Inter-arrival times

Definition 3.

Let

- N Poisson process with intensity λ

We define $T_0 = 0$ and

$$\begin{aligned} T_n &= \inf\{t \geq 0;\, N(t) = n\} \\ X_n &= T_n - T_{n-1} \end{aligned}$$

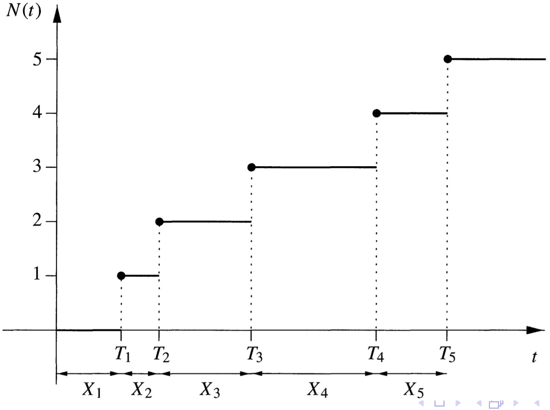
Then X_n is called inter-arrival time

From X to N

N as a function of X: We have

$$T_n = \sum_{i=1}^{n} X_i$$

$$N(t) = \max\{n \geq 0;\ T_n \leq t\}$$



Distribution of the inter-arrival times

Theorem 4.

Let

- N Poisson process with intensity λ
- $\{X_j;\, j \geq 1\}$ inter-arrival times

Then

The X_j's are i.i.d with common distribution $\mathcal{E}(\lambda)$

Proof of Theorem 4 (1)

Variable X_1: We have

$$\mathbf{P}\left(X_1 > t\right) = \mathbf{P}\left(N(t) = 0\right) = \exp(-\lambda\, t)$$

Thus

$$X_1 \sim \mathcal{E}(\lambda)$$

Proof of Theorem 4 (2)

Conditioning on X_1: Write

$$\begin{aligned}
&\mathbf{P}\left(X_2 > t \middle| X_1 = t_1\right) \\
=&\mathbf{P}\left(\text{No arrival in } (t_1, t_1 + t] \middle| X_1 = t_1\right) \\
=&\mathbf{P}\left(N(t_1, t_1 + t]) = 0 \middle| N(t_1) = 1,\ X_1 = t_1\right) \\
=&\exp(-\lambda\, t)
\end{aligned}$$

Thus

$$X_2 \sim \mathcal{E}(\lambda), \quad \text{and} \quad X_2 \perp\!\!\!\perp X_1$$

Proof of Theorem 4 (3)

Conditioning on X_n: Write $\tau = \sum_{i=1}^{n} t_i$ and

$$
\begin{aligned}
& \mathbf{P}\left(X_{n+1} > t \,\middle|\, X_1 = t_1, \ldots, X_n = t_n\right) \\
=& \mathbf{P}\left(\text{No arrival in } (\tau, \tau + t] \,\middle|\, X_1 = t_1, \ldots, X_n = t_n\right) \\
=& \mathbf{P}\left(N(\tau, \tau + t] = 0 \,\middle|\, N(\tau) = n,\, X_1 = t_1, \ldots, X_n = t_n\right) \\
=& \exp(-\lambda\, t)
\end{aligned}
$$

Thus

$$X_{n+1} \sim \mathcal{E}(\lambda), \quad \text{and} \quad X_{n+1} \perp\!\!\!\perp (X_1, \ldots, X_n)$$

Another proof of $N(t) \sim \mathcal{P}(\lambda t)$

Strategy:

1. Start from $\{X_k;\ k \geq 1\}$ inter-arrival times
2. Set $T_n = \sum_{k=1}^{n} X_k$
3. If X_k's are i.i $\mathcal{E}(\lambda)$ random variables, then $T_n \sim \Gamma(\lambda, n)$
4. Compute

$$\begin{aligned}
\mathbf{P}\left(N(t) = j\right) &= \mathbf{P}\left(T_j \leq t < T_{j+1}\right) \\
&= \mathbf{P}\left(T_j \leq t\right) - \mathbf{P}\left(T_{j+1} \leq t\right) \\
&= \frac{(\lambda t)^j}{j!} \exp(-\lambda\, t)
\end{aligned}$$

Outline

Definition of birth process

Definition 5.

Let

- $N = \{N(t);\ t \geq 0\}$ process with $N(0) = 0$ and $N(t) \in \mathbb{N}$

Then N is a birth process if

- $N(0) = 0$ and $t \mapsto N(t)$ is $\nearrow$
- Probability $\mathbf{P}\left(N(t+h) = n+m \,|\, N(t) = n\right)$ of the form

$$\begin{cases} \lambda_n h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda_n h + o(h) & \text{if } m = 0 \end{cases}$$

- Conditional on $N(s)$, $N(t) - N(s) \perp\!\!\!\perp$ values of N on $[0, s]$

Remark and particular case

Interpretation: For a birth process
$\hookrightarrow$ the birth rate depends on the population size

Poisson case:

When $\lambda_n = \lambda$, i.e birth rate independent of the population size

Simple birth

Model:

- Living individuals give birth independently of one another
- Each individual gives birth with probability $\lambda h + o(h)$
- No death

Claim:

The simple birth process is a birth process with $\lambda_n = n\,\lambda$

Simple birth (2)

Justification of the claim: Let $M = \#$ births in $(t, t+h)$. Then

$$\mathbf{P}\left(M = n + m \middle| N(t) = n\right) = \binom{n}{m} (\lambda h)^m (1 - \lambda h)^{n-m} + o(h)$$

$$= \begin{cases} n\lambda h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - n\lambda h + o(h) & \text{if } m = 0 \end{cases}$$

Simple birth with immigration

Model:

- Living individuals give birth independently of one another
- Each individual gives birth with probability $\lambda h + o(h)$
- No death
- Constant immigration ν

Form of λ_n: We get

$$\lambda_n = n\,\lambda + \nu$$

Forward ode's for the probabilities

Proposition 6.

Let

- N birth process
- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$

Set

$$p_{ij}(t) = \mathbf{P}\left(N(s+t) = j \middle| N(s) = i\right)$$

Then for $j \geq i$ the function p_{ij} satisfies

$$p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t)\,,$$

with initial condition $p_{ij}(0) = \delta_{ij}$

Proof of Proposition 6 (1)

Conditioning on a small interval: We have

$$
\begin{aligned}
& p_{ij}(t+h) \\
&= \mathbf{P}\left(N(t+h) = j \,|\, N(0) = i\right) \\
&= \sum_{k \in S} \mathbf{P}\left(N(t+h) = j,\, N(t) = k \,|\, N(0) = i\right) \\
&= \sum_{k \in S} \mathbf{P}\left(N(t+h) = j \,|\, N(0) = i,\, N(t) = k\right) \mathbf{P}\left(N(t) = k \,|\, N(0) = i\right) \\
&= \sum_{k \in S} \mathbf{P}\left(N(t+h) = j \,|\, N(t) = k\right) \mathbf{P}\left(N(t) = k \,|\, N(0) = i\right) \\
&= (1 - \lambda_j h)\, p_{ij}(t) + (\lambda_{j-1} h)\, p_{i,j-1}(t) + o(h)
\end{aligned}
$$

Proof of Proposition 6 (2)

Recall:

$$p_{ij}(t+h) - p_{ij}(t) = \left(\lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{ij}(t)\right) h + o(h)$$

Differentiating: We end up with a system of ode's

$$p'_{ij}(t) = \lambda_{j-1}\, p_{i,j-1}(t) - \lambda_j\, p_{ij}(t)$$

Initial condition:

$$p_{ij}(0) = \delta_{ij} \equiv \mathbf{1}_{(i=j)}$$

Backward ode's

Proposition 7.

Let

- N birth process
- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$

Set

$$p_{ij}(t) = \mathbf{P}\left(N(s+t) = j \,\middle|\, N(s) = i\right)$$

Then for $j \geq i$ the function p_{ij} satisfies

$$p'_{i,j}(t) = \lambda_i p_{i+1,j}(t) - \lambda_i p_{i,j}(t)\,,$$

with initial condition $p_{ij}(0) = \delta_{ij}$

Proof of Proposition 7 (1)

Backward conditioning on a small interval: We have

$$
\begin{aligned}
& p_{ij}(t+h) \\
&= \mathbf{P}\left(N(t+h)=j \mid N(0)=i\right) \\
&= \sum_{k\in S} \mathbf{P}\left(N(t+h)=j,\, N(h)=k \mid N(0)=i\right) \\
&= \sum_{k\in S} \mathbf{P}\left(N(t+h)=j \mid N(0)=i,\, N(h)=k\right) \mathbf{P}\left(N(h)=k \mid N(0)=i\right) \\
&= \sum_{k\in S} \mathbf{P}\left(N(t+h)=j \mid N(h)=k\right) \mathbf{P}\left(N(h)=k \mid N(0)=i\right) \\
&= p_{ij}(t)\left(1-\lambda_i h\right) + p_{i+1,j}(t)\left(\lambda_i h\right) + o(h)
\end{aligned}
$$

Proof of Proposition 7 (2)

Recall:

$$p_{ij}(t+h) - p_{ij}(t) = \left(\lambda_i p_{i+1,j}(t) - \lambda_i p_{ij}(t)\right) h + o(h)$$

Differentiating: We end up with a system of ode's

$$p'_{ij}(t) = \lambda_i \, p_{i+1,j}(t) - \lambda_i \, p_{ij}(t)$$

Initial condition:

$$p_{ij}(0) = \delta_{ij} \equiv \mathbf{1}_{(i=j)}$$

Solving the forward system

Theorem 8.

Let

- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$
- Set of indices $\{0 \leq i, j < \infty\}$

Then the system of equations

- $p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t)$ if $j \geq i$
- $p_{ij}(0) = \delta_{ij}$
- $p_{ij}(t) = 0$ if $j < i$

admits a unique solution

Proof of Theorem 8

Case $i = j$: The equation becomes

$$p'_{i,i}(t) = -\lambda_i p_{i,i}(t), \quad \text{initial condition} \quad p_{i,i}(0) = 1$$

Thus

$$p_{i,i}(t) = \exp(-\lambda_i t)$$

General case:

Obtained by recursion

Laplace transform

Definition: Let $f : \mathbb{R}_+ \to \mathbb{R}$. Then

$$\mathcal{L}f(s) = \hat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Possible strategy to solve a differential equation:

1. Transform diff. equation into algebraic problem in s variable.
2. Solve algebraic problem and find $\hat{f}$.
3. Invert Laplace transform and find f.

Existence of Laplace transform

Theorem 9.

Hypothesis:

- f piecewise continuous on $[0, A]$ for each $A > 0$.
- $|f(t)| \leq Ke^{at}$ for $K \geq 0$ and $a \in \mathbb{R}$.

Conclusion:
$\mathcal{L}f(s)$ exists for $s > a$.

Vocabulary: f satisfying $|f(t)| \leq Ke^{at}$
$\hookrightarrow$ Called function of exponential order.

Table of Laplace transforms

Function f	Laplace transform $\hat{f}$	Domain of $\hat{f}$
$\mathbf{1}$	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\mathbf{1}_{[0,1)}(t) + k\,\mathbf{1}_{(t=1)}$	$\frac{1-e^{-s}}{s}$	$s > 0$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \|a\|$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \|a\|$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

Table of Laplace transforms (2)

Function f	Laplace transform $\hat{f}$	Domain of $\hat{f}$
$t^n e^{at},\ n \in \mathbb{N}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t) f(t-c)$	$e^{-cs}\hat{f}(s)$	
$e^{ct} f(t)$	$\hat{f}(s-c)$	
$f(ct),\ c > 0$	$\frac{1}{c}\hat{f}(\frac{s}{c})$	
$\int_0^t f(t-\tau) g(\tau)$	$\hat{f}(s)\hat{g}(s)$	
$\delta(t-c)$	e^{-cs}	
$f^{(n)}(t)$	$s^n \hat{f}(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	
$(-t)^n f(s)$	$\hat{f}^{(n)}(s)$	

Linearity of Laplace transform

Example of function f:

$$f(t) = 5\, e^{-2t} - 3\, \sin(4t).$$

Laplace transform by linearity: we find

$$\begin{aligned} \mathcal{L}f(s) &= 5\left[\mathcal{L}(e^{-2t})\right](s) - 3\left[\mathcal{L}(\sin(4t)\right](s) \\ &= \frac{5}{s+2} - \frac{12}{s^2+16}. \end{aligned}$$

Interest of Laplace transform

Laplace:

- 1749-1827, lived in France
- Mostly mathematician
- Called the French Newton
- Contributions in
  - Mathematical physics
  - Analysis, partial differential equations
  - Celestial mechanics
  - Probability (central limit theorem)

General interest of Laplace transform:

In many branches of mathematics (analysis - geometry - probability)

Interest for differential equations:

Deal with impulsive (discontinuous) forcing terms.

Relation between $\mathcal{L}f$ and $\mathcal{L}f'$

Theorem 10.

Hypothesis:

1. f continuous, f' piecewise continuous on $[0, A]$
   $\hookrightarrow$ for each $A > 0$.
2. $|f(t)| \leq Ke^{at}$ for $K, a \geq 0$.

Conclusion: $\mathcal{L}f'$ exists and

$$\mathcal{L}f'(s) = s\mathcal{L}f(s) - f(0)$$

Proof of Theorem 10

Integration by parts:

$$\int_0^A e^{-st} f'(t)\,dt = \left[e^{-st} f(t)\right]_0^A + s \int_0^A e^{-st} f(t)\,dt$$

Laplace transform of transitions

Proposition 11.

Let

- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$
- Set of indices $\{0 \leq i, j < \infty\}$
- p_{ij} solution to forward system
  $p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t)$

Then for $i \leq j$ the Laplace transform $\hat{p}_{ij}$ satisfies

$$\hat{p}_{ij}(s) = \frac{1}{\lambda_j} \prod_{\ell=i}^{j} \frac{\lambda_\ell}{s + \lambda_\ell}$$

Proof of Proposition 13 (1)

Laplace transform of the forward equation: The equation

$$p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t)$$

becomes

$$s\, \hat{p}_{ij}(s) - \delta_{ij} = \lambda_{j-1} \hat{p}_{i,j-1}(s) - \lambda_j \hat{p}_{ij}(s)$$

Rearranging terms: We get

$$(s + \lambda_j)\, \hat{p}_{ij}(s) = \delta_{ij} + \lambda_{j-1} \hat{p}_{i,j-1}(s)$$

Proof of Proposition 13 (2)

Case $j > i$: Since $\delta_{ij} = 0$ in that case, we get

$$\begin{aligned}
\hat{p}_{ij}(s) &= \frac{\lambda_{j-1}}{s+\lambda_j}\,\hat{p}_{i,j-1}(s) \\
&= \frac{\lambda_{j-1}}{s+\lambda_j}\,\frac{\lambda_{j-2}}{s+\lambda_{j-1}}\,\hat{p}_{i,j-2}(s) \\
&= \frac{1}{\lambda_j}\frac{\lambda_j}{s+\lambda_j}\,\frac{\lambda_{j-1}}{s+\lambda_{j-1}}\,\lambda_{j-2}\,\hat{p}_{i,j-2}(s)
\end{aligned}$$

Conclusion: Iterating the above computation, we get

$$\hat{p}_{ij}(s) = \frac{1}{\lambda_j}\prod_{\ell=i}^{j}\frac{\lambda_\ell}{s+\lambda_\ell}$$

Backward and forward system

Proposition 12.

Consider the backward system

$$\pi'_{i,j}(t) = \lambda_i \pi_{i+1,j}(t) - \lambda_i \pi_{i,j}(t)\,, \tag{1}$$

Then

The solution $\{p_{ij};\ i,j \geq 0\}$ to the forward system also solves the system (1)

Proof of Proposition 15

Backward equation in Laplace mode: We get

$$(s + \lambda_i)\,\hat{\pi}_{ij}(s) = \delta_{ij} + \lambda_i \hat{\pi}_{i+1,j}(s) \tag{2}$$

Forward solves backward: Take

$$\hat{\pi}_{ij}(s) = \hat{p}_{ij}(s) = \frac{1}{\lambda_j} \prod_{\ell=i}^{j} \frac{\lambda_\ell}{s + \lambda_\ell}$$

This solves (2)

Problem with the backward system

Main problem:
Backward system may not have a unique solution

Minimal solution:

The unique solution of the forward system
is a minimal solution of the backward system

Minimal solution of the backward system

Proposition 13.

Let

- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$
- Set of indices $\{0 \leq i, j < \infty\}$
- p_{ij} solution to forward system
  $p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t)$

Then

$$\pi_{ij} \text{ solution of the backward system}$$
$$\Longrightarrow \text{We have } p_{i,j}(t) \leq \pi_{i,j}(t) \text{ for all } i, j \in S \text{ and } t \geq 0$$

Backward system and explosion

Relating explosion time and uniqueness:

1. If $\sum_{j\in S} p_{i,j}(t) = 1$, then
   $\hookrightarrow$ $p_{i,j}$ is the unique solution of the backward system
2. Problem: $\{p_{i,j}(t);\, j \in S\}$ is not always a distribution
3. This is related to explosion time: we might have

$$\mathbf{P}\left(T_\infty < \infty\right) > 0, \quad \text{where} \quad T_\infty = \lim_{n\to\infty} T_n$$

Honest birth process

Definition 14.

Let

- N birth process
- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$
- $\{T_n;\, n \geq 1\}$ arrival times

Then N is said to be honest if

$$\mathbf{P}\left(T_\infty = \infty\right) = 1$$

Sum of exponential random variables

Proposition 15.

Let

- $\{X_n; n \geq 1\}$ sequence of independent random variables
- Each X_n is such that $X_n \sim \mathcal{E}(\lambda_{n-1})$
- $T_\infty = \sum_{n=1}^{\infty} X_n$

Then

$$\mathbf{P}\left(T_\infty < \infty\right) = \begin{cases} 0, & \text{if } \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty \end{cases}$$

Proof of Proposition 15 (1)

Case $\sum_{n\geq 1} \lambda_n^{-1} < \infty$: Using Fubini-Tonelli we have

$$\mathbf{E}\left[T_\infty\right] = \mathbf{E}\left[\sum_{n=1}^{\infty} X_n\right] = \sum_{n=1}^{\infty} \frac{1}{\lambda_{n-1}} < \infty$$

Thus

$$\mathbf{P}\left(T_\infty < \infty\right) = 0$$

Proof of Proposition 15 (2)

Case $\sum_{n\geq 1} \lambda_n^{-1} = \infty$, strategy: We have

$$\mathbf{E}\left[e^{-T_\infty}\right] = 0 \implies \mathbf{P}\left(e^{-T_\infty} = 0\right) = 1$$
$$\implies \mathbf{P}\left(T_\infty = \infty\right) = 1$$

We will thus prove

$$\mathbf{E}\left[e^{-T_\infty}\right] = 0$$

Proof of Proposition 15 (3)

Case $\sum_{n\geq1}\lambda_n^{-1}=\infty$, computation: We have

$$
\begin{aligned}
\mathbf{E}\left[e^{-T_\infty}\right] &= \mathbf{E}\left[\prod_{n=1}^{\infty} e^{-X_n}\right] \\
&= \lim_{N\to\infty} \mathbf{E}\left[\prod_{n=1}^{N} e^{-X_n}\right] \qquad \text{(monotone convergence)} \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} \mathbf{E}\left[e^{-X_n}\right] \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} \frac{1}{1+\lambda_{n-1}^{-1}} = \left(\prod_{n=1}^{\infty}\left(1+\frac{1}{\lambda_{n-1}}\right)\right)^{-1}
\end{aligned}
$$

Proof of Proposition 15 (4)

Infinite products: If $u_n \geq 0$, then

$$\prod_{n=1}^{\infty}(1+u_n) = \infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} u_n = \infty \tag{3}$$

Pseudo-proof of (3): We have

$$\ln\left(\prod_{n=1}^{\infty}(1+u_n)\right) = \sum_{n=1}^{\infty} \ln(1+u_n) \asymp \sum_{n=1}^{\infty} u_n$$

Proof of Proposition 15 (5)

Recall: We have seen

$$\mathbf{E}\left[e^{-T_\infty}\right] = \left(\prod_{n=1}^{\infty}\left(1+\frac{1}{\lambda_{n-1}}\right)\right)^{-1}$$

Application of (3):

$$\mathbf{E}\left[e^{-T_\infty}\right] \iff \prod_{n=1}^{\infty}\left(1+\frac{1}{\lambda_{n-1}}\right) = \infty \iff \sum_{n\geq 1}\lambda_n^{-1} = \infty$$

Conclusion:

$$T_\infty = \infty \quad \iff \quad \sum_{n\geq 1}\lambda_n^{-1} = \infty$$

Application to birth process

Proposition 16.

Let

- N birth process
- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$
- $\{T_n;\, n \geq 1\}$ arrival times

Then N is honest iff

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$$

Final remarks

Notes before next section:

1. Poisson and birth processes are Markov processes
   $\hookrightarrow$ Due to $(N(t) - N(s)) \perp\!\!\!\perp$ Past, given $N(s) = i$
2. They are in fact strong Markov processes
   $\hookrightarrow$ Definition to be seen later
3. Problems can occur due to explosions
   $\hookrightarrow$ This could not be observed in discrete time

Outline

1. Birth processes and the Poisson process
   - Poisson process
   - Birth processes
2. Continuous time Markov chain
   - General definitions and transitions
   - Generators
   - Classification of states

Outline

Vocabulary

Stochastic process:

- Family $\{X(t);\, t \in [0, \infty)\}$ of random variables
- Family evolving in a random but prescribed manner
- Here $X(t) \in S$, where S countable state space with $N = |S|$

Markov evolution:

Conditioned on $X(t)$,
the evolution does not depend on the past

Markov chain

Definition 17.

Let

- $X = \{X(t);\ t \geq 0\}$ stochastic process

We say that X is a continuous time Markov chain if

$$\begin{aligned} &\mathbf{P}\left(X(t_n) = j \,|\, X(t_1) = i_1, \ldots, X(t_{n-1}) = i_{n-1}\right) \\ =&\mathbf{P}\left(X(t_n) = j \,|\, X(t_{n-1}) = i_{n-1}\right), \end{aligned}$$

for all

- $0 \leq t_1 < \cdots < t_n < \infty$
- $i_1, \ldots, i_n, j \in S$

Differences with discrete time

Main difference:

- No time unit
- Therefore no exact analogue of P

Method 1:

- Use infinitesimal calculus
- This leads to infinitesimal generator

Method 2:

- Embedded chain

Birth process as Markov process

Proposition 18.

Let

- N birth process
- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$

Then

N is a Markov process

Proof of Proposition 18 (1)

Setting: Consider

- $s_1 < \cdots < s_n < s < t$
- $i_1, \ldots, i_n, j \in S$

Aim: Prove

$$\begin{aligned} &\mathbf{P}\left(N(t) = j \,|\, N(s_1) = i_1, \ldots, N(s_n) = i_n, N(s) = i\right) \\ =&\mathbf{P}\left(N(t) = j \,|\, N(s) = i\right) \end{aligned}$$

Equivalent statement: Prove that

$$\begin{aligned} &\mathbf{P}\left(N(t) - N(s) = j - i \,|\, N(s_1) = i_1, \ldots, N(s_n) = i_n, N(s) = i\right) \\ =&\mathbf{P}\left(N(t) - N(s) = j - i \,|\, N(s) = i\right) \end{aligned}$$

Proof of Proposition 18 (2)

Recall: We wish to prove

$$\mathbf{P}\left(N(t) - N(s) = j - i \,|\, N(s_1) = i_1, \ldots, N(s_n) = i_n, N(s) = i\right)$$
$$= \mathbf{P}\left(N(t) - N(s) = j - i \,|\, N(s) = i\right)$$

Defining some sets: Consider

- $A_{st} = (N(t) - N(s) = j - i)$
- $B_{s_1,\ldots,s_n} = N(s_1) = i_1, \ldots, N(s_n) = i_n$
- $C_s = (N(s) = i)$

Rephrasing our claim: Now we wish to prove

$$\mathbf{P}\left(A_{st} \,|\, B_{s_1,\ldots,s_n} \cap C_s\right) = \mathbf{P}\left(A_{st} \,|\, C_s\right)$$

Proof of Proposition 18 (3)

General formula: We have

$$\mathbf{P}\left(A_{st} \cap B_{s_1,\ldots,s_n} \middle| C_s\right) = \mathbf{P}\left(A_{st} \middle| B_{s_1,\ldots,s_n} \cap C_s\right) \mathbf{P}\left(B_{s_1,\ldots,s_n} \middle| C_s\right) \qquad (4)$$

Conditional independence: In Definition 5 we had the assumption

$$\text{Conditional on } N(s),\ N(t) - N(s) \perp\!\!\!\perp \text{ values of } N \text{ on } [0, s]$$

This reads

$$\mathbf{P}\left(A_{st} \cap B_{s_1,\ldots,s_n} \middle| C_s\right) = \mathbf{P}\left(A_{st} \middle| C_s\right) \mathbf{P}\left(B_{s_1,\ldots,s_n} \middle| C_s\right) \qquad (5)$$

Conclusion: Combining (4) and (5) we end up with

$$\mathbf{P}\left(A_{st} \middle| B_{s_1,\ldots,s_n} \cap C_s\right) = \mathbf{P}\left(A_{st} \middle| C_s\right)$$

Transition probabilities

Definition 19.

Let X be a continuous-time Markov chain. Then

1. The transition probabilities are given by

$$p_{ij}(s,t) = \mathbf{P}\left(X(t) = j \,|\, X(s) = i\right) \quad \text{for} \quad s < t, \; i,j \in S$$

2. X is homogeneous if for all n, i, j we have

$$p_{ij}(s,t) = p_{ij}(0, t-s) \equiv p_{ij}(t-s)$$

Hypothesis 20.

In the chapter we always assume that X is homogeneous

Transitions for the Poisson process

Proposition 21.

Let

- N Poisson process
- Intensity λ

Then N is homogeneous and

$$p_{ij}(s,t) = p_{ij}(t-s) = \exp(-\lambda(t-s))\,\frac{(\lambda(t-s))^{j-i}}{(j-i)!}$$

Proof of Proposition 21

Expression for the conditional probabilities: We have

$$
\begin{aligned}
p_{ij}(s,t) &= \mathbf{P}\left(N(t)=j \,|\, N(s)=i\right) \\
&= \mathbf{P}\left(N(t)-N(s)=j-i \,|\, N(s)=i\right) \\
&= \mathbf{P}\left(N(t)-N(s)=j-i\right) \quad (N(t)-N(s) \perp\!\!\!\perp \mathrm{N(s)}) \\
&= \mathbf{P}\left(N(t-s)=j-i\right) \quad \text{(Homogeneity)} \\
&= \exp(-\lambda(t-s))\,\frac{(\lambda(t-s))^{j-i}}{(j-i)!} \quad \text{(Poisson distribution)}
\end{aligned}
$$

Transition semigroup

Definition 22.

Let X be a homogeneous Markov chain. Then

1. We set
$$P_t = (p_{ij}(t))_{i,j\in S}$$
2. The family
$$\{P_t;\ t \geq 0\}$$
is called transition semigroup

Stochastic semigroup

Theorem 23.

The family P is a stochastic semigroup, that is

1. $P_0 = \mathrm{Id}$
2. For all $t \geq 0$, P_t is a stochastic matrix, i.e
   - $p_{ij}(t) \geq 0$, for all i, j
   - $\sum_j p_{ij}(t) = 1$, for all i
3. Chapman-Kolmogorov holds true:

$$P_{s+t} = P_s P_t$$

Proof of Theorem 23

Proof of item 2: For $t \geq 0$ we have

$$\begin{aligned}\sum_{j \in S} p_{ij}(t) &= \sum_{j \in S} \mathbf{P}\left(X(t) = j \middle| X(0) = i\right) \\ &= \mathbf{P}\left(\cup_{j \in S} X(t) = j \middle| X(0) = i\right) \\ &= 1\end{aligned}$$

Proof of Theorem 23 (2)

Proof of item 3: For $s, t \geq 0$ we have

$$
\begin{aligned}
p_{ij}(s+t) &= \mathbf{P}(X(s+t) = j \mid X(0) = i) \\
&= \sum_k \mathbf{P}(X(s+t) = j,\, X(s) = k \mid X(0) = i) \\
&= \sum_k \mathbf{P}(X(s+t) = j \mid X(s) = k,\, X(0) = i)\mathbf{P}(X(s) = k \mid X(0) = i) \\
&= \sum_k \mathbf{P}(X(s+t) = j \mid X(s) = k)\mathbf{P}(X(s) = k \mid X(0) = i) \\
&= \sum_k p_{ik}(s) p_{kj}(t)
\end{aligned}
$$

Standard semigroup

Definition 24.

Let

- X Markov chain with transition P

Then P is said to be standard if

$$\lim_{t\to 0} P_t = \mathrm{Id},$$

that is

$$\lim_{t\to 0} p_{ij}(t) = \delta_{ij}, \text{ for all } i, j \in S$$

Outline

Continuity of standard semigroups

Proposition 25.

Assume

- X Markov chain
- The transition P is standard

Then P is continuous: for all $t \geq 0$ we have

$$\lim_{h \to 0} P_{t+h} = P_t,$$

that is

$$\lim_{h \to 0} p_{ij}(t+h) = p_{ij}(t), \text{ for all } i, j \in S$$

Behavior close to 0

Taylor expansions: We have (admitted)

$$\begin{aligned} p_{ij}(h) &= g_{ij}h + o(h) \\ p_{ii}(h) &= 1 + g_{ii}h + o(h) \end{aligned}$$

Signs of g_{ij}: If we want $p_{ij}(h) \in [0,1]$ we need

$$g_{ij} \geq 0, \quad \text{and} \quad g_{ii} \leq 0$$

Meaning of g_{ij}'s

Interpretation: Starting from $X(t) = i$,

1. Nothing happens with probability

$$1 + g_{ii}h + o(h)$$

2. The chain jumps from i to j with probability

$$g_{ij}h + o(h)$$

Terminology:

The matrix $G = (g_{ij})_{i,j\in S}$ is called generator of the Markov chain

Basic property of the generator

Proposition 26.

Assume

- X Markov chain
- The transition P is standard
- There is a generator G

Then for most cases we have

$$\sum_{j \in S} g_{ij} = 0, \quad \text{for all} \quad i \in S$$

Generator for birth process

Proposition 27.

Let

- N birth process
- Intensities $\{\lambda_j;\, j \geq -1\}$, with $\lambda_{-1} = 0$

Then the generator G of N is given by

$$g_{ii} = -\lambda_i, \quad g_{i,i+1} = \lambda_i, \quad g_{ij} = 0 \text{ otherwise}, \tag{6}$$

that is

$$G = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ 0 & -\lambda_1 & \lambda_1 & 0 & 0 & \cdots \\ 0 & 0 & -\lambda_2 & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Proof of Proposition 27

Expansion for birth transitions: We have seen (cf Definition 5)

$$
\begin{aligned}
p_{n,n}(t, t+h) &= p_{n,n}(h) = 1 - \lambda_n h + o(h) \\
p_{n,n+1}(t, t+h) &= p_{n,n+1}(h) = \lambda_n h + o(h) \\
p_{n,j}(t, t+h) &= p_{n,j}(h) = o(h), \quad \text{if } j \geq n+2
\end{aligned}
$$

General expansion: We have also seen the general expression

$$
\begin{aligned}
p_{nn}(h) &= 1 + g_{nn} h + o(h) \\
p_{nj}(h) &= g_{nj} h + o(h)
\end{aligned}
$$

Conlusion:
We easily get (6) by identification

Matrix form of the generator

Proposition 28.

Assume

- X Markov chain
- The transition P is standard

Then we have

$$\lim_{h\to 0}\frac{1}{h}\left(P_h - \mathrm{Id}\right) = G\,,$$

that is

$$\lim_{h\to 0}\frac{1}{h}\left(p_{ij}(h) - \delta_{ij}\right) = g_{ij}, \text{ for all } i,j\in S$$

Proof of Proposition 28

Main argument: Rephrasing of

$$
\begin{aligned}
p_{ij}(h) &= g_{ij} h + o(h) \\
p_{ii}(h) &= 1 + g_{ii} h + o(h)
\end{aligned}
$$

Transitions from generator: forward equations

Proposition 29.

Assume

- X Markov chain
- The transition P is standard

Then P_t satisfies the differential equation

$$P_t' = P_t \, G \, .$$

that is

$$p_{ij}'(t) = \sum_{k \in S} p_{ik}(t) g_{kj}, \text{ for all } i, j \in S$$

Proof of Proposition 29

Application of Chapman-Kolmogorov:

$$\begin{aligned} p_{ij}(t+h) &= \sum_{k\in S} p_{ik}(t) p_{kj}(h) \\ &\simeq p_{ij}(t)\,(1+g_{jj}h) + \sum_{k\neq j} p_{ik}(t) g_{kj} h \\ &= p_{ij}(t) + \sum_{k\in S} p_{ik}(t) g_{kj} h \end{aligned}$$

Differentiating:

$$\frac{1}{h}\left(p_{ij}(t+h) - p_{ij}(t)\right) \simeq \sum_{k\in S} p_{ik}(t) g_{kj} = (P_t G)_{ij}$$

Transitions from generator: matrix exponential

Proposition 30.

Assume

- X Markov chain
- The transition P is standard

Then P_t satisfies the relation

$$P_t = e^{t\,G}\,, \quad \text{where} \quad e^{t\,A} \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

General inter-arrival

Proposition 31.

Let

- X Markov chain with transition P_t
- U random variable defined by

$$U = \inf\{t \geq 0; X(s+t) \neq i\}$$

Then we have

$$\mathcal{L}(U|\, X(s) = i) = \mathcal{E}(-g_{ii}),$$

that is

$$\mathbf{P}(U > t|\, X(s) = i) = \exp(-g_{ii}\, t)$$

Proof of Proposition 31 (1)

Properties of exponential random variables: If $Z \sim \mathcal{E}(\mu)$, then

$$\mathbf{P}\left(Z > a + b \middle| Z > a\right) = \mathbf{P}\left(Z > b\right) = \exp\left(-\mu\, b\right) \qquad (7)$$

Remarks about (7):

1. Relation (7) can be interpreted as lack of memory
2. It can also be interpreted as no aging
3. In fact (7) characterizes the distribution $\mathcal{E}(\mu)$

Proof of Proposition 31 (2)

Main argument: We have

$$
\begin{aligned}
&\mathbf{P}\left(U > a+b \middle|\, U > a,\, X(s)=i\right) \\
&= \mathbf{P}\left(U > a+b \middle|\, X(s+a)=i,\, X(s)=i\right) \\
&= \mathbf{P}\left(a + U\circ\theta_a > a+b \middle|\, X(s+a)=i,\, X(s)=i\right) \\
&= \mathbf{P}\left(U\circ\theta_a > b \middle|\, X(s+a)=i\right) \qquad \text{(Markov)} \\
&= \mathbf{P}\left(U\circ\theta_a > b \middle|\, X(s)\circ\theta_a=i\right) \\
&= \mathbf{P}\left(U > b \middle|\, X(s)=i\right) \qquad \text{(Homogeneity)}
\end{aligned}
$$

Imbedded Markov chain

Proposition 32.

Let

- X Markov chain with standard transition P_t
- Assume $X(0) = i$

Then we have

$$\mathbf{P}\left(X \text{ jumps to } j \middle| X(0) = i\right) = -\frac{g_{ij}}{g_{ii}}$$

Proof of Proposition 32

Argument on a small interval: On $[t, t+h)$,

$$\begin{aligned}
\mathbf{P}\left(X \text{ jumps to } j \,|\, X \text{ jumps}\right) &\simeq \frac{p_{ij}(h)}{1 - p_{ii}(h)} \\
&\simeq \frac{g_{ij}\, h}{(-g_{ii}\, h)} \\
&\simeq -\frac{g_{ij}}{g_{ii}}
\end{aligned}$$

Example with 2 states (1)

Model: We consider

- State space $S = \{1, 2\}$
- Generator

$$G = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

Example with 2 states (2)

Pathwise description: Applying Propositions 31 and 32 we get

1. If X is in state 1 then
   - X stays at 1 an amount of time $\sim \mathcal{E}(\alpha)$
   - Next X jumps to 2
2. If X is in state 2 then
   - X stays at 2 an amount of time $\sim \mathcal{E}(\beta)$
   - Next X jumps to 1

Example with 2 states (3)

Forward equation: Can be read as

$$\begin{bmatrix} p'_{11}(t) & p'_{12}(t) \\ p'_{21}(t) & p'_{22}(t) \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

Sub-system for p_{11}, p_{12}: We get a separate system of the form

$$\begin{bmatrix} p'_{11}(t) \\ p'_{12}(t) \end{bmatrix} = A \begin{bmatrix} p_{11}(t) \\ p_{12}(t) \end{bmatrix}, \quad \text{with} \quad A = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}$$

Example with 2 states (3)

Eigenvalue decomposition for A: We get

$$\lambda_1 = 0, \quad \text{with} \quad \mathbf{v}_1 = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

$$\lambda_2 = -(\alpha + \beta), \quad \text{with} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

General form of the solution: We get

$$\begin{bmatrix} p_{11}(t) \\ p_{12}(t) \end{bmatrix} = c_1 \begin{bmatrix} \beta \\ \alpha \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp\left(-(\alpha + \beta)t\right)$$

Example with 2 states (4)

Computation of constants: We use

$$\lim_{t\to\infty} (p_{11}(t) + p_{12}(t)) = 1, \quad \text{and} \quad p_{12}(0) = 0$$

and we get

$$c_1 = \frac{1}{\alpha+\beta}, \quad \text{and} \quad c_2 = -\frac{\alpha}{\alpha+\beta}$$

Unique solution: We end up with

$$\begin{bmatrix} p_{11}(t) \\ p_{12}(t) \end{bmatrix} = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} - \frac{\alpha}{\alpha+\beta} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp\left(-(\alpha+\beta)t\right)$$

Example with 2 states (5)

Sub-system for p_{21}, p_{22}: We get a separate system of the form

$$\begin{bmatrix} p'_{21}(t) \\ p'_{22}(t) \end{bmatrix} = A \begin{bmatrix} p_{21}(t) \\ p_{22}(t) \end{bmatrix}, \quad \text{with} \quad A = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}$$

Unique solution: We end up with

$$\begin{bmatrix} p_{21}(t) \\ p_{22}(t) \end{bmatrix} = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} - \frac{\beta}{\alpha+\beta} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp\left(-(\alpha+\beta)t\right)$$

Outline

Irreducibility of chains

Proposition 33.

Let X Markov chain with standard transition P_t

Then we have

1. For every pair $i, j \in S$, either
$$p_{ij}(t) = 0 \text{ for all } t > 0$$
or
$$p_{ij}(t) > 0 \text{ for all } t > 0$$
2. Terminology: if $p_{ij}(t) > 0$ for all $t > 0$
$\hookrightarrow$ X is said to be irreducible
3. In order to know if X is irreducible
$\hookrightarrow$ draw graph related to G

Birth process example

Recall: For the birth process,

$$G = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ 0 & -\lambda_1 & \lambda_1 & 0 & 0 & \cdots \\ 0 & 0 & -\lambda_2 & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Nature of states:

All states are transient

2 states example

Recall:

$$G = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

Nature of states:

The chain is irreducible

Stationary distribution

Definition 34.

Let

- X Markov chain with transition P
- π vector

Then π is a stationary distribution if

1. $\pi_j \geq 0$ for all $j \in S$ and $\sum_{j \in S} \pi_j = 1$
2. π satisfies $\pi = \pi P_t$ for all $t \geq 0$, that is

$$\pi_j = \sum_{i \in S} \pi_i p_{ij}(t), \quad \text{for all} \quad j \in S$$

Interpretation of stationary distribution

Proposition 35.

Let

- X Markov chain with transition P
- π invariant distribution

Then

$$X_0 \sim \pi \quad \Longrightarrow \quad X(t) \sim \pi \quad \text{for all } t \geq 0$$

Otherwise stated,

$$\mathbf{P}\left(X(t) = j \,|\, X(0) \sim \pi\right) = \pi_j$$

Stationary distribution and generator

Proposition 36.

Let

- X Markov chain with transition P and generator G
- π distribution

Then

$$\pi \text{ invariant distribution} \quad \Longleftrightarrow \quad \pi G = 0$$

Proof of Proposition 36

Basic relation: We have

$$\pi G = 0 \iff \pi G^n = 0$$

Reasoning with matrix exponential: We get

$$\begin{aligned}
\pi G = 0 &\iff \sum_{n=1}^{\infty} \frac{t^n}{n!} \pi G^n = 0, \quad \text{for all } t \geq 0 \\
&\iff \pi \sum_{n=1}^{\infty} \frac{t^n}{n!} G^n = 0, \quad \text{for all } t \geq 0 \\
&\iff \pi \sum_{n=0}^{\infty} \frac{t^n}{n!} G^n = \pi, \quad \text{for all } t \geq 0 \\
&\iff \pi P_t = \pi, \quad \text{for all } t \geq 0
\end{aligned}$$

Ergodic theorem

Proposition 37.

Let

- X Markov chain with transition P and generator G
- Assume X is irreducible

Then

1. If there exists a stationary distribution π, then
$$\pi \text{ is unique and } \lim_{t\to\infty} p_{ij}(t) = \pi_j \text{ for all } i,j \in S$$
2. If there is no stationary distribution π, then
$$\lim_{t\to\infty} p_{ij}(t) = 0 \text{ for all } i,j \in S$$

2 states example (1)

Recall:

$$G = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

Invariant distribution: The chain is irreducible and we have

$$\pi = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix} \quad \Longrightarrow \quad \pi G = 0$$

2 states example (2)

Recall: We have seen

$$\begin{bmatrix} p_{11}(t) \\ p_{12}(t) \end{bmatrix} = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} - \frac{\alpha}{\alpha+\beta} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp\left(-(\alpha+\beta)t\right)$$

Verifying the ergodic theorem: We get

$$\lim_{t\to 0} \begin{bmatrix} p_{11}(t) \\ p_{12}(t) \end{bmatrix} = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}$$