Discrete time Markov chains

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Elements of Stochastic Processes – MA 532

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by Grimmett-Stirzaker

Outline

Outline

Vocabulary

Stochastic process:

- Family $\{X_n;\ n \geq 0, n \text{ integer}\}$ of random variables
- Family evolving in a random but prescribed manner
- Here $X_n \in S$, where S countable state space with $N = |S|$

Discrete time:

- In this chapter we consider X indexed by $n \in \mathbb{N}$, discrete
- Later continuous time, $\{X_t;\ t \geq 0\}$

Markov evolution:

Conditioned on X_n,
the evolution does not depend on the past

Markov chain

Definition 1.

Let

- $X = \{X_n;\ n \geq 0, n \text{ integer}\}$ stochastic process

We say that X is a Markov chain if

$$\mathbf{P}\left(X_n = s \middle| X_0 = x_0, \ldots, X_{n-1} = x_{n-1}\right) = \mathbf{P}\left(X_n = s \middle| X_{n-1} = x_{n-1}\right),$$

for all $n \geq 1$ and $x_0, \ldots, x_{n-1}, s \in S$

Random walk as a Markov chain

Proposition 2.

Let

- $X_1, \ldots, X_n$ Bernoulli random variables with values ± 1,

$$\mathbf{P}(X_i = 1) = p, \qquad \mathbf{P}(X_i = -1) = 1 - p$$

- The X_i's are independent
- The random walk defined by $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i$$

Then

S is a Markov chain

Proof of Proposition 2

Decomposition for S_n: We write

$$S_{n+1} = S_n + X_{n+1}$$

Conditional probability: We have

$$\begin{aligned}
&\mathbf{P}\left(S_{n+1} = s \,|\, S_0 = x_0, \ldots, S_n = x_n\right) \\
&= \mathbf{P}\left(S_n + X_{n+1} = s \,|\, S_0 = x_0, \ldots, S_n = x_n\right) \\
&= \mathbf{P}\left(X_{n+1} = s - x_n \,|\, S_0 = x_0, \ldots, S_n = x_n\right) \\
&= \mathbf{P}\left(X_{n+1} = s - x_n\right) \\
&= \mathbf{P}\left(S_{n+1} = s \,|\, S_n = x_n\right)
\end{aligned}$$

This proves the Markov property

Alternative formulations for Markov's property

Proposition 3.

The Markov property is equivalent to any of the following:

1. For all $n_1 < n_2 < \cdots < n_k \leq n$ we have

$$\mathbf{P}\left(X_n = s \middle| X_{n_1} = x_{n_1}, \ldots, X_{n_k} = x_{n_k}\right) = \mathbf{P}\left(X_n = s \middle| X_{n_k} = x_{n_k}\right)$$

2. For all $m, n \geq 0$,

$$\mathbf{P}\left(X_{m+n} = s \middle| X_0 = x_0, \ldots, X_m = x_m\right) = \mathbf{P}\left(X_{m+n} = s \middle| X_m = x_m\right)$$

Transition probability

Reduction to $S \subset \mathbb{N}$:

- Recall that $X_n \in S$
- S countable $\Longrightarrow$ S in one-to-one correspondence with $S' \subset \mathbb{N}$
- We denote $(X_n = x_i)$ by $(X_n = i)$

Important quantity to describe X: Transition probability, defined by

$$\mathbf{P}\left(X_{n+1} = j \,|\, X_n = i\right)$$

It depends on n, i, j

Andrey Markov

Andrey Markov's life:

- Lifespan: 1856-1922, $\simeq$ St Petersburg
- Not a very good student
  $\hookrightarrow$ except in math
- Contributions in analysis and probability
- Used chains for
  $\hookrightarrow$ appearance of vowels
- Professor in St Petersburg
  - Suspended after 1908 students riots
  - Resumed teaching in 1917

Fact: More than 50 mathematical objects named after Markov!!

Homogeneous Markov chains

Definition 4.

Let X be a Markov chain. Then

1. X is homogeneous if for all n, i, j we have

$$\mathbf{P}\left(X_{n+1} = j \middle| X_n = i\right) = \mathbf{P}\left(X_1 = j \middle| X_0 = i\right)$$

2. If X is homogeneous we define a transition matrix

$$P = (p_{ij}) \quad \text{with} \quad p_{ij} = \mathbf{P}\left(X_{n+1} = j \middle| X_n = i\right)$$

Hypothesis 5.

In the chapter we always assume that X is homogeneous

Stochastic matrix

Theorem 6.

The matrix P is stochastic, that is

1. $p_{ij} \geq 0$, for all i, j
2. $\sum_j p_{ij} = 1$, for all i

n-step transition

Definition 7.

Let X be a Markov chain. We set

$$P(m, m+n) = \left(p_{ij}(m, m+n)\right)_{i,j} ,$$

with

$$p_{ij}(m, m+n) = P\left(X_{m+n} = j \,|\, X_m = i\right)$$

Remark:

- P describes the short term behavior of X
- $P(m, m+n)$ describes the long term behavior of X

Chapman-Kolmogorov equations

Theorem 8.

Let X be a Markov chain with transition p. Then

1. For $m, n, r \geq 0$ we have
$$p_{ij}(m, m+n+r) = \sum_k p_{ik}(m, m+n) p_{kj}(m+n, m+n+r)$$
2. As a matrix,
$$P(m, m+n+r) = P(m, m+n) P(m+n, m+n+r)$$
3. In particular,
$$P(m, m+n) = P^n$$

Proof of Theorem 8 (1)

Preliminary identity:

$$\mathbf{P}(A \cap B|\, C) = \mathbf{P}(A|\, B \cap C)\mathbf{P}(B|\, C)$$

Proof: Start from right hand side,

$$\begin{aligned}
\mathbf{P}(A|\, B \cap C)\mathbf{P}(B|\, C) &= \frac{\mathbf{P}(A \cap B \cap C)}{\mathbf{P}(B \cap C)} \frac{\mathbf{P}(B \cap C)}{\mathbf{P}(C)} \\
&= \frac{\mathbf{P}\left((A \cap B) \cap C\right)}{\mathbf{P}(C)} \\
&= \mathbf{P}(A \cap B|\, C)
\end{aligned}$$

Proof of Theorem 8 (2)

Computation: We have

$$\begin{aligned}
p_{ij}(m, m+n+r) &= \mathbf{P}(X_{m+n+r} = j \,|\, X_m = i) \\
&= \sum_k \mathbf{P}(X_{m+n+r} = j,\, X_{m+n} = k \,|\, X_m = i) \\
&= \sum_k \mathbf{P}(X_{m+n+r} = j \,|\, X_{m+n} = k,\, X_m = i)\mathbf{P}(X_{m+n} = k \,|\, X_m = i) \\
&= \sum_k \mathbf{P}(X_{m+n+r} = j \,|\, X_{m+n} = k)\mathbf{P}(X_{m+n} = k \,|\, X_m = i) \\
&= \sum_k p_{ik}(m, m+n)p_{kj}(m+n, m+n+r)
\end{aligned}$$

Law of X_n

Proposition 9.

Consider the row vector

$$\mu_i^{(n)} = \mathbf{P}(X_n = i)$$

Then

$$\mu^{(m+n)} = \mu^{(m)} P^n$$

In particular,

$$\mu^{(n)} = \mu^{(0)} P^n$$

Proof of Proposition 9

Computation: Write

$$
\begin{aligned}
\mu_j^{(m+n)} &= \mathbf{P}\left(X_{m+n} = j\right) \\
&= \sum_i \mathbf{P}\left(X_{m+n} = j \middle| X_m = i\right) \mathbf{P}\left(X_m = i\right) \\
&= \sum_i \mu_i^{(m)} p_{ij}(m, m+n) \\
&= \left[\mu^{(m)} P^n\right]_j
\end{aligned}
$$

Example: weather in West Lafayette (1)

Model: We choose $S = \{1, \ldots, 6\} := \{VN, N, SN, SG, G, VG\}$.

Transition: from empirical data, we have found

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.4 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.8 & 0 & 0.2 \end{pmatrix}$$

Example: weather in West Lafayette (2)

Model: We choose $S = \{1, \ldots, 6\} := \{VN, N, SN, SG, G, VG\}$.

Prediction for J+2:

$$P^2 = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0.24 & 0.76 & 0 & 0 & 0 & 0 \\ 0.12 & 0.3 & 0.16 & 0.19 & 0.18 & 0.05 \\ 0 & 0 & 0 & 0.44 & 0.21 & 0.35 \\ 0 & 0 & 0 & 0.55 & 0.35 & 0.1 \\ 0 & 0 & 0 & 0.4 & 0.56 & 0.04 \end{pmatrix}$$

Example: weather in West Lafayette (3)

Model: We choose $S = \{1, \dots, 6\} := \{VN, N, SN, SG, G, VG\}$.

Prediction for J+28:

$$P^{28} = \begin{pmatrix} 0.29 & 0.71 & 0 & 0 & 0 & 0 \\ 0.29 & 0.71 & 0 & 0 & 0 & 0 \\ 0.14 & 0.36 & 7.2 \times 10^{-12} & 0.23 & 0.16 & 0.10 \\ 0 & 0 & 0 & 0.47 & 0.33 & 0.20 \\ 0 & 0 & 0 & 0.47 & 0.33 & 0.20 \\ 0 & 0 & 0 & 0.47 & 0.33 & 0.20 \end{pmatrix}$$

Easy criteria to establish Markov property

Proposition 10.

Let X be a process such that

- $X_{n+1} = \varphi(X_n, Z_{n+1})$
- $Z_{n+1} \perp\!\!\!\perp (X_0, \ldots, X_n)$
- $\{Z_n;\, n \geq 1\}$ i.i.d family
- φ is a given fixed function

Then

1. X is a Markov chain
2. The transition is given by

$$p_{ij} = \mathbf{P}\left(\varphi(i, Z_1) = j\right)$$

Simple random walk case (1)

State space:

$$S = \mathbb{Z}$$

Markov property: We have seen

- $X_{n+1} = X_n + Z_{n+1} = \varphi(X_n, Z_{n+1})$
- $\varphi(x, y) = x + y$
- $\{Z_n;\ n \geq 1\}$ i.i.d family
- $\mathbf{P}(Z_1 = 1) = p$ and $\mathbf{P}(Z_1 = -1) = q$

Thus

X is a Markov chain

Simple random walk case (2)

Transition probability: We have

$$
\begin{aligned}
p_{ij} &= \mathbf{P}\left(i + Z_1 = j\right) \\
&= \mathbf{P}\left(Z_1 = j - i\right) \\
&= \begin{cases} p, & \text{if } j = i + 1 \\ q, & \text{if } j = i - 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

Simple random walk case (3)

Expression for X_n: Starting from i, write

$$X_n = i + \sum_{k=1}^{n} Z_k$$

Relation with Bernoulli random variables: We have

$$Z_k = 2Y_k - 1, \quad \text{with} \quad Z_k \sim \mathcal{B}(p)$$

Thus

$$X_n = i + 2\sum_{k=1}^{n} Y_k - n$$

Simple random walk case (4)

n-step transition: We obtain

$$X_n = j \iff \sum_{k=1}^{n} Y_k = \frac{1}{2}(n + j - i)$$

Thus

$$p_{ij}(n) = \begin{cases} \binom{n}{\frac{1}{2}(n+j-i)} p^{\frac{1}{2}(n+j-i)} q^{\frac{1}{2}(n-j+i)}, & \text{if } n + j - i \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

Conditions on i, j:

- $-n \le j - i \le n$
- $j - i$ has the same parity as n

Branching process case (1)

State space:

$$S = \mathbb{N}$$

Markov property: We have seen

- $X_{n+1} = \sum_{k=1}^{X_n} Z_k^{(n+1)} = \varphi(X_n, \mathbf{Z}^{(n+1)})$
- $\mathbf{Z}^{(n)} = \{\mathbf{Z}_k^{(n)};\ k \geq 1\}$ is a sequence
- $\varphi(x, \mathbf{z}) = \sum_{k=1}^{x} z_k$
- $\{\mathbf{Z}^{(n)};\ n \geq 1\}$ i.i.d family
  $\hookrightarrow$ with $(Z_k^{(n)})_{k\geq 1}$ i.i.d with common pgf G

Thus

X is a Markov chain

Branching process case (2)

Transition probability: We have

$$\begin{aligned} p_{ij} &= \mathbf{P}\left(\sum_{k=1}^{i} Z_k^{(1)} = j\right) \\ &= \frac{1}{j!} \times \text{Coefficient of } s^j \text{ in } (G(s))^i \end{aligned}$$

n-step transition: We obtain

$$p_{ij}(n) = \frac{1}{j!} \times \text{Coefficient of } s^j \text{ in } (G_n(s))^i$$

Outline

Questions about Markov chains

Main questions

1. Does the MC X_n go to ∞ when $n \to \infty$?
2. Does it return to state i after $n = 0$?
3. How often does it return to i?
4. What is the range of $X_n(\omega)$?

Methodologies to answer those questions

1. We have seen: pgf's for random walks and branching
2. Now: Markov chain methods

Persistent and transient states

Definition 11.

Let

- X Markov chain
- i state in S

Then

1. i is called persistent or recurrent if

$$\mathbf{P}\left(X_n = i \text{ for some } n \geq 1 \middle| X_0 = i\right) = 1$$

2. i is called transient if

$$\mathbf{P}\left(X_n = i \text{ for some } n \geq 1 \middle| X_0 = i\right) < 1$$

First passage time probabilities

Definition 12.

Let

- X Markov chain and i, j states in S

Then we define

1. Probability that
   $\hookrightarrow$ 1st visit to j starting from i takes place at step n:

$$f_{ij}(n) = \mathbf{P}\left(X_1 \neq j, \ldots, X_{n-1} \neq j, X_n = j \middle| X_0 = i\right)$$

2. Probability that X ever visits j starting from i:

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$$

Alternative definition for $f_{ij}(n)$

First visit to j: We set $T_j = \infty$ if there is no visit to j, and

$$T_j = \inf \{n \geq 1;\, X_n = j\}$$

Expression for $f_{ij}(n)$: We have

$$\begin{aligned} f_{ij}(n) &= \mathbf{P}\left(X_1 \neq j, \ldots, X_{n-1} \neq j, X_n = j \middle| X_0 = i\right) \\ &= \mathbf{P}\left(T_j = n \middle| X_0 = i\right) \end{aligned}$$

Some pgf's

Pgf's P and F: We set

$$P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}(n) s^n, \qquad F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}(n) s^n$$

Remarks:

1. Conventions above: $p_{ij}(0) = \delta_{ij}$ and $f_{ij}(0) = 0$
2. i persistent iff $f_{ii} = 1$
3. For $|s| < 1$, the series $P_{ij}(s)$ and $F_{ij}(s)$ are convergent
4. $P_{ij}(1)$ and $F_{ij}(1)$ are defined through Abel's theorem
5. $f_{ij} = F_{ij}(1)$

Relation between F and P

Theorem 13.

Let X_n be a Markov chain with transition p. Then

1. P_{ii} and F_{ii} satisfy

$$P_{ii}(s) = 1 + F_{ii}(s) P_{ii}(s)$$

2. For $i \neq j$, the function P_{ij} verifies

$$P_{ij}(s) = F_{ij}(s) P_{jj}(s)$$

Proof of Theorem 13 (1)

Events: We set

$$A_m = (X_m = j), \qquad B_k = (T_j = k)$$

Decomposition for A_m: We have

$$A_m = A_m \cap \left(\bigcup_{k=1}^{n} B_k \right) = \bigcup_{k=1}^{n} (A_m \cap B_k)$$

Proof of Theorem 13 (2)

Preliminary identity: Recall that

$$\mathbf{P}(A \cap B | C) = \mathbf{P}(A | B \cap C)\mathbf{P}(B | C)$$

Decomposition for probabilities: We get

$$\begin{aligned} \mathbf{P}(A_m \cap B_k | X_0 = i) &= \mathbf{P}\left(A_m | B_k, X_0 = i\right) \mathbf{P}(B_k | X_0 = i) \\ &\stackrel{\text{Markov}}{=} \mathbf{P}\left(A_m | X_k = j\right) \mathbf{P}(B_k | X_0 = i) \qquad (1) \end{aligned}$$

Proof of Theorem 13 (3)

Convolution relation: Equation (1) can be read as

$$\begin{aligned} p_{ij}(m) &= \mathbf{P}(A_m |\, X_0 = i) \\ &= \sum_{k=1}^{n} \mathbf{P}(A_m \cap B_k |\, X_0 = i) \\ &= \sum_{k=1}^{n} p_{jj}(m-k) f_{ij}(k), \quad \text{for} \quad m \geq 1, \quad \text{and} \quad p_{ij}(0) = \delta_{ij} \end{aligned}$$

Expression with generating functions: We get

$$P_{ij}(s) - \delta_{ij} = F_{ij}(s) P_{jj}(s)$$

Criterion for recurrence and transience

Proposition 14.

Let X_n be a Markov chain with transition p. Then

1. If $\sum_{n=0}^{\infty} p_{jj}(n) = \infty$, then
   - State j is persistent
   - $\sum_{n=0}^{\infty} p_{ij}(n) = \infty$ for all i's such that $f_{ij} > 0$
2. If $\sum_{n=0}^{\infty} p_{jj}(n) < \infty$, then
   - State j is transient
   - $\sum_{n=0}^{\infty} p_{ij}(n) < \infty$ for all i

Proof of Proposition 14 (1)

Expression for $P_{jj}(s)$: From Theorem 13 we have

$$P_{jj}(s) = \frac{1}{1 - F_{jj}(s)}, \quad \text{for} \quad |s| < 1$$

Limit as $s \nearrow 1$: We get

- $P_{jj}(s) \to \infty$ iff $F_{jj}(1) = 1$
- $F_{jj}(1) = f_{jj}$
- j persistent iff $f_{jj} = 1$

Thus

$$j \text{ persistent iff } \lim_{s \nearrow 1} P_{jj}(s) = \infty$$

Proof of Proposition 14 (2)

Recall: We have seen

$$j \text{ persistent iff } \lim_{s\nearrow 1} P_{jj}(s) = \infty$$

Application of Abel:

$$\lim_{s\nearrow 1} P_{jj}(s) = \sum_{n=0}^{\infty} p_{jj}(n)$$

Conclusion:

$$j \text{ persistent iff } \sum_{n=0}^{\infty} p_{ij}(n) = \infty$$

Proof of Proposition 14 (3)

Another relation for $p_{ij}(n)$: We have seen

$$P_{ij}(s) = F_{ij}(s) P_{jj}(s)$$

Taking limits $s \nearrow 1$ we get

$$\sum_{n=0}^{\infty} p_{ij}(n) = f_{ij} \sum_{n=0}^{\infty} p_{jj}(n)$$

Conclusion: If $\sum_{n=0}^{\infty} p_{jj}(n) = \infty$, then

$\sum_{n=0}^{\infty} p_{ij}(n) = \infty$ for all i's such that $f_{ij} > 0$

Behavior of $p_{ij}(n)$

Proposition 15.

Let

- X Markov chain with transition p
- j transient state

Then

$$\lim_{n\to\infty} p_{ij}(n) = 0$$

Simple random walk case

Proposition 16.

Let

- X simple random walk
- Parameters p and $q = 1 - p$

Then

$$X \text{ is persistent iff } p = \frac{1}{2}$$

Proof of Proposition 16 (1)

Formula for $p_{jj}(m)$: According to (26),

$$p_{jj}(2n) = \binom{2n}{n}(pq)^n, \qquad p_{jj}(2n+1) = 0$$

Stirling's formula:

$$m! \equiv \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

Equivalent for $p_{jj}(2n)$: We get, as $n \to \infty$,

$$p_{jj}(2n) \sim \frac{(4pq)^n}{(\pi n)^{1/2}}$$

Proof of Proposition 16 (2)

Recall: We have seen that

$$p_{jj}(2n) \sim \frac{(4pq)^n}{(\pi n)^{1/2}}$$

Case $p = \frac{1}{2}$: We get

$$p_{jj}(2n) \sim \frac{1}{(\pi n)^{1/2}}$$

Thus

$$\sum_{n=0}^{\infty} p_{jj}(2n) = \infty \quad \Longrightarrow \quad \text{State } j \text{ persistent}$$

Proof of Proposition 16 (3)

Recall: We have seen that

$$p_{jj}(2n) \sim \frac{(4pq)^n}{(\pi n)^{1/2}}$$

Case $p \neq \frac{1}{2}$: We get

$$p_{jj}(2n) \sim \frac{(c_p)^n}{(\pi n)^{1/2}}, \quad \text{with} \quad c_p < 1$$

Thus

$$\sum_{n=0}^{\infty} p_{jj}(2n) < \infty \quad \Longrightarrow \quad \text{State } j \text{ transient}$$

Number of visits

Recall: We have seen that

State j is either persistent or transient

Number of visits: We set

$$N(i) = \# \text{ times that } X \text{ visits its starting point } i$$

Fact: We have

$$\mathbf{P}\left(N(i) = \infty \,|\, X_0 = i\right) = \begin{cases} 1\,, & \text{if } i \text{ persistent} \\ 0\,, & \text{if } i \text{ transient} \end{cases}$$

Behavior of T_j for a transient state

Recall: We set $T_j = \infty$ if there is no visit to j, and

$$T_j = \inf \{n \geq 1;\, X_n = j\}$$

Mean for T_j if j is transient: Whenever j is transient,

$$\begin{aligned} \mathbf{P}\left(T_j = \infty \middle| X_0 = j \right) &> 0 \\ \mathbf{E}\left[T_j \middle| X_0 = j \right] &= \infty \end{aligned}$$

Mean recurrence time

Definition 17.

Let

- X Markov chain
- i state in S

Then we set

$$\mu_i = \mathbf{E}\left[T_i \,|\, X_0 = i\right] = \begin{cases} \sum_{n=1}^{\infty} n\, f_{ii}(n)\,, & \text{if } i \text{ is persistent} \\ \infty\,, & \text{if } i \text{ is transient} \end{cases}$$

Null and positive states

Definition 18.

Let

- X Markov chain
- i persistent state in S, with mean recurrence time μ_i

Then

1. i is said to be null if $\mu_i = \infty$
2. i is said to be positive if $\mu_i < \infty$

Criterion for nullity

Theorem 19.

Let

- X Markov chain
- i persistent state in S

Then

$$i \text{ is null iff } \lim_{n\to\infty} p_{ii}(n) = 0$$

Period

Definition 20.

Let

- X Markov chain, i state in S

Then

1. The period of i is given by

$$d(i) = \gcd\{n;\ p_{ii}(n) > 0\}$$

2. The state i is aperiodic if $d(i) = 1$, periodic if $d(i) > 1$

Interpretation: The period describes
$\hookrightarrow$ Times at which returns to i are possible

Ergodic states

Definition 21.

Let

- X Markov chain
- i state in S

Then i is said to be ergodic if

$$i \text{ is persistent, positive and aperiodic}$$

Simple random walk case

Proposition 22.

Let

- X simple random walk
- Parameters p and $q = 1 - p$

Then the states are

1. Periodic with period 2
2. Transient if $p \neq \frac{1}{2}$
3. Null persistent if $p = \frac{1}{2}$

Proof of Proposition 22 (1)

Transience if $p \neq \frac{1}{2}$:

This has been established in Proposition 16

Null recurrence if $p = \frac{1}{2}$:

- This has been established
  $\hookrightarrow$ in Generating functions - Proposition 12
- We have seen that $\mathbf{E}[T_0] = \infty$

Proof of Proposition 22 (2)

Another way to look at null recurrence: If $p = \frac{1}{2}$ we have seen

$$p_{ii}(2n) \sim \frac{1}{(\pi n)^{1/2}}, \qquad p_{ii}(2n+1) = 0$$

Hence

$$\lim_{n\to\infty} p_{ii}(n) = 0$$

According to Theorem 19, i is recurrent null

Period 2: The fact that $d(i) = 2$ stems from

$$p_{ii}(2n) > 0, \qquad p_{ii}(2n+1) = 0$$

Branching process case

Proposition 23.

Consider a branching process with

- $Z_1 \sim f$, f with pgf G
- $\mathbf{P}(Z_1 = 0) = f(0) > 0$

Then

1. 0 is an absorbing state:

$$\mathbf{P}(X_n = 0 \text{ for all } n | \, X_0 = i) = 1$$

2. Other states are transient

Proof of Proposition 23

Proof:

Done in Exercise 2

Outline

Communication

Recall: For a Markov chain X, we have seen that

$$\mathbf{P}\left(X_n = j \middle| X_0 = i\right) = p_{ij}(n)$$

Communication:
We say that i communicates with j if

There exists $n \geq 0$ such that $\mathbf{P}\left(X_n = j \middle| X_0 = i\right) = p_{ij}(n) > 0$.

Notation: $i \to j$.

Intercommunication

Intercommunication:
If $i \to j$ and $j \to i$, we say that i and j intercommunicate.
Notation: $i \leftrightarrow j$.

Remarks:

1. For all $i \in S$, we have $i \leftrightarrow i$, since $p^0(i,i) = 1$.
2. If $i \to j$ and $j \to k$, then $i \to k$.

Graph related to a Markov chain

Definition 24.

Let X be a Markov chain with transition p.
We define a graph $\mathcal{G}(X)$ given by

- $\mathcal{G}(X)$ is an oriented graph
- The vertices of $\mathcal{G}(X)$ are points in S.
- The edges of $\mathcal{G}(X)$ are given by the set

$$\mathbb{V} \equiv \{(i,j);\ i \neq j,\ p(i,j) > 0\}.$$

Example

Definition of the chain: Take $S = \{1, 2, 3, 4, 5\}$ and

$$p = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

Related graph: to be done in class

Graph and communication

Proposition 25.

Let X be a Markov chain with transition p. Then

$$i \to j$$
iff
$$i = j \text{ or there exists an oriented path from } i \text{ to } j \text{ in } \mathcal{G}(X)$$

Proof of Proposition 25

Relation with the graph: If $i \neq j$ we have

$(i \to j) \Leftrightarrow$ There exists $n \geq 1$ such that $p_{ij}(n) > 0$

$\Leftrightarrow$ There exists $n \geq 1$ such that

$$\sum_{i_1,\ldots,i_{n-1} \in E} p_{i,i_1} \cdots p_{i_{n-1},j} > 0$$

$\Leftrightarrow$ There exists $n \geq 1$ and $i_1, \ldots, i_{n-1} \in E$ such that

$$p_{i,i_1} \cdots p_{i_{n-1},j} > 0$$

$\Leftrightarrow$ There exists an oriented path from i to j in $\mathcal{G}(X)$

Irreducible classes

Proposition 26.

Let

- X Markov chain with transition p

Then

1. The relation $\leftrightarrow$ is an equivalence relation.
2. Denote $C_1, \ldots, C_l$ the equivalence classes for $\leftrightarrow$ in S. Then $\rightarrow$ is a partial order relation between classes:
$$C_1 \rightarrow C_2 \text{ and } C_2 \rightarrow C_3 \Longrightarrow C_1 \rightarrow C_3$$
3. $C_1 \rightarrow C_2$ iff $\exists\ i \in C_1$ and $j \in C_2$ such that $i \rightarrow j$.
4. The classes are called irreducible

Example (1)

Definition of the chain: Take $E = \{1, 2, 3, 4, 5\}$ and

$$p = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

Example (2)

Recall: We have $E = \{1, 2, 3, 4, 5\}$ and

$$p = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

Related classes:
$C_1 = \{1, 3\}$, $C_2 = \{2\}$ and $C_3 = \{4, 5\}$.
We have $C_2 \to C_1$

Nature of intercommunicating states

Theorem 27.

Let

- X Markov chain with transition p
- i, j such that $i \leftrightarrow j$

Then

1. i, j have the same period
2. i transient iff j transient
3. i null persistent iff j null persistent

Proof of Theorem 27 – item 2 (1)

A positive quantity: If $i \leftrightarrow j$, then there exists $m, n \geq 1$ such that

$$\alpha \equiv p_{ij}(m) p_{ji}(n) > 0$$

Application of Chapman-Kolmogorov: We get

$$p_{ii}(m + r + n) \geq p_{ij}(m) p_{jj}(r) p_{ji}(n) = \alpha\, p_{jj}(r)$$

Summing over r: We get

$$\sum_{r=0}^{\infty} p_{ii}(r) < \infty \quad \Longrightarrow \quad \sum_{r=0}^{\infty} p_{jj}(r) < \infty$$

Proof of Theorem 27 – item 2 (2)

Conclusion:

$$i \text{ transient} \quad \Longrightarrow \quad j \text{ transient}$$

Closed class

Definition 28.

An equivalent class C is closed if:

$$\text{For all } i \in C \text{ and } j \notin C, \text{ we have } i \not\to j.$$

Some rules for closedness:

- If there exists a unique class C, it is closed
- There exists a unique closed class C
  $\Leftrightarrow$ There exists a class C s.t for all $i \in E$, we have $i \to C$.

Example ctd (1)

Definition of the chain: Take $E = \{1, 2, 3, 4, 5\}$ and

$$p = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

Example ctd (2)

Recall: The related classes are
$C_1 = \{1, 3\}$, $C_2 = \{2\}$ and $C_3 = \{4, 5\}$.
We have $C_2 \to C_1$

Closed classes: We find

$$C_1, C_3 \text{ closed, and } C_2 \text{ not closed}$$

Random walk example

Proposition 29.

Let

- X simple random walk
- Parameters p and $q = 1 - p$

Then

1. There is a unique class, $C = \mathbb{Z}$
2. This class is closed
3. If one state is transient, all the states are transient
4. If one state is null pers., all the states are null pers.
5. All the states have the same period

Decomposition theorem

Theorem 30.

Let

- X Markov chain with transition p
- S state space

Then S can be partitioned uniquely as

$$S = T \cup C_1 \cup C_2 \cup \cdots ,$$

where

- $T \equiv$ Set of transient states
- $C_k \equiv$ irreducible closed class of persistent states

Finite state space case

Proposition 31.

Let

- X Markov chain with transition p
- S finite state space with $S = T \cup C_1 \cup C_2 \cup \cdots$

Then

1. At least 1 state in S is persistent
2. All persistent states are positive
3. Later we will see: every state in C_k is positive persistent

Proof of Proposition 31

Recall: We have seen in Proposition 15 that

$$j \text{ transient state} \quad \Longrightarrow \quad \lim_{n\to\infty} p_{ij}(n) = 0$$

Assume all states are transient: Then for $i \in C_k$,

$$\lim_{n\to\infty} \sum_{j\in C_k} p_{ij}(n) = 0$$

Contradiction: If C_k is closed,

$$\lim_{n\to\infty} \sum_{j\in C_k} p_{ij}(n) = 1$$

Example ctd (1)

Definition of the chain: Take $E = \{1, 2, 3, 4, 5\}$ and

$$p = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

Example ctd (2)

Recall: The related classes are
$C_1 = \{1, 3\}$, $C_2 = \{2\}$ and $C_3 = \{4, 5\}$.
C_1, C_3 closed, and C_2 not closed

Information about the classes: We find

All states in C_1, C_3 (closed class) are positive persistent
State 2 in C_2 transient

Outline

Outline

Stationary distribution

Definition 32.

Let

- X Markov chain with matrix transition P
- π vector

Then π is a stationary distribution if

1. $\pi_j \geq 0$ for all $j \in S$ and $\sum_{j \in S} \pi_j = 1$
2. π satisfies $\pi = \pi P$, that is

$$\pi_j = \sum_{i \in S} \pi_i p_{ij}, \quad \text{for all} \quad j \in S$$

Interpretation of stationary distribution

Proposition 33.

Let

- X Markov chain with matrix transition P
- π invariant distribution

Then

$$X_0 \sim \pi \quad \Longrightarrow \quad X_n \sim \pi \quad \text{for all } n \geq 0$$

Otherwise stated,

$$\mathbf{P}\left(X_n = j \,|\, X_0 \sim \pi\right) = \pi_j$$

Proof of Proposition 33

Distribution of X_1: We have

$$\begin{aligned}
\mathbf{P}\left(X_1 = j \,|\, X_0 \sim \pi\right) &= \sum_{i \in S} \mathbf{P}\left(X_1 = j \,|\, X_0 = i\right) \pi_i \\
&= (\pi P)_j \\
&= \pi_j
\end{aligned}$$

Distribution of X_n: Use a recursion and

$$\mathbf{P}\left(X_{n+1} = j\right) = \sum_{i \in S} \mathbf{P}\left(X_{n+1} = j \,|\, X_n = i\right) \mathbf{P}\left(X_n = i\right)$$

Stationary distributions and persistent chains

Theorem 34.

Let

- X Markov chain with matrix transition P
- X irreducible

Then

$$X \text{ has a stationary distribution} \iff \text{All states are non-null persistent}$$

Stationary distributions and return times

Theorem 35.

Let

- X Markov chain with matrix transition P
- X irreducible
- X admits a stationary distribution π

Then

$$\pi_i = \frac{1}{\mu_i} = \frac{1}{\mathbf{E}[T_i | X_0 = i]}$$

Hints about the proof

Main ingredient: Prove that

$$\mu_k = \sum_{i \in S} \rho_i(k), \quad \text{with} \quad \rho_i(k) = \sum_{n=1}^{\infty} \mathbf{P}\left(X_n = i, T_k \geq n \middle| X_0 = k\right),$$

is solution to $\mu = \mu P$

Idea for $\pi_i = (\mu_i)^{-1}$: One writes

$$\pi_i = \text{"Average time spent at } i\text{"} \simeq \frac{1}{\text{"Average time to return at } i\text{"}}$$

Example (1)

Definition of the chain: Take $S = \{1, 2, 3, 4\}$ (hence $|S| < \infty$) and

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Example (2)

Related classes:
$C_1 = \{1\}, \ C_2 = \{2, 3, 4\}$
$\hookrightarrow$ C_1 closed C_2 non closed.

Partial conclusion: C_1 transient, at least one recurrent state in C_2.

Invariant measure:
Solve the system $\pi = \pi P$ and $\langle \pi, \mathbf{1} \rangle = 1$. We find

$$\pi = (0, \, 1/4, \, 1/2, \, 1/4) \, .$$

Conclusion: All states in C_2 are non-null persistent

Example (3)

Remark:

- It is almost always easier to solve the system

$$\pi = \pi\, p \quad \text{and} \quad \langle \pi, \mathbf{1} \rangle = 1$$

  than to compute $\mathbf{E}_i[T_i]$

- However, in the current case a direct computation is possible

Example (4)

Direct analysis: We find

- $\mathbf{E}_1[T_1] = \infty$ since 1 is transient
- $\mathbf{E}_3[T_3] = 2$ since $T_3 = 2$ under $\mathbf{P}_3$.
- In order to compute $\mathbf{E}_2[T_2]$:

$$\begin{aligned}
\mathbf{E}_2\left[\mathbf{1}_{(T_2>2k+2)}\right] &= \mathbf{E}_2\left[\mathbf{1}_{(T_2>2k)}\,\mathbf{1}_{(T_2>2k+2)}\right] \\
&= \mathbf{E}_2\left\{\mathbf{1}_{(T_2>2k)}\,\mathbf{E}_{X_{2k}}\left[\mathbf{1}_{(T_2(A^{2k})>2)}\right]\right\} \\
&= \mathbf{E}_2\left\{\mathbf{1}_{(T_2>2k)}\,\mathbf{E}_4\left[\mathbf{1}_{(T_2(A^{2k})>2)}\right]\right\} \\
&= \mathbf{E}_2\left[\mathbf{1}_{(T_2>2k)}\,p_{4,3}\,p_{3,4}\right] = \frac{1}{2}\mathbf{E}_2\left[\mathbf{1}_{(T_2>2k)}\right]
\end{aligned}$$

We deduce $\mathbf{P}_2(T_2 > 2k) = 1/2^k$ and $\mathbf{E}_2[T_2] = 4 = \mathbf{E}_4[T_4]$.

Criterion for positivity/nullity

Theorem 36.

Let

- X Markov chain with matrix transition P
- X irreducible
- X recurrent

Then

1. There exists a measure x satisfying $x = x\,P$
2. x is unique up to multiplicative constant
3. x has strictly positive entries
4. The chain is positive if $\sum_{i \in S} x_i < \infty$
5. The chain is null if $\sum_{i \in S} x_i = \infty$

Criterion for transience

Theorem 37.

Let

- X Markov chain with matrix transition P
- X irreducible
- s any state in S

Then

$$X \text{ is transient}$$
$$\Longleftrightarrow$$
$$\text{There exists a non zero solution } \{y_i;\, i \neq s\}$$
$$\text{to } y_i = \sum_{j \neq s} p_{ij} y_j, \text{ with } |y_i| \leq 1$$

Random walk with retaining barrier (1)

Model: Random walk on $\mathbb{N}$
$\hookrightarrow$ With retaining barrier at 0

Transition probability: We get

$$p_{00} = q, \quad p_{i,i+1} = p, \text{ if } i \geq 0, \quad p_{i,i-1} = q, \text{ if } i \geq 1$$

Notation: We set

$$\rho = \frac{p}{q}$$

Random walk with retaining barrier (2)

Proposition 38.

Let X be the random walk with retaining barrier. Then

1. If $p > \frac{1}{2}$, the chain is transient
2. If $p < \frac{1}{2}$, the chain is non-null persistent
   $\hookrightarrow$ with stationary distribution given by

$$\pi = \text{Nbin}(1, 1 - \rho)$$

3. If $p = \frac{1}{2}$, the chain is null persistent

Proof of Proposition 38 (1)

Case $q < p$: One verifies that

$$y_i = 1 - \rho^{-i} \quad \text{solves} \quad y_i = \sum_{j \neq s} p_{ij} y_j$$

Thus X transient

Case $q > p$: One sees that

$$\pi = \text{Nbin}(1, 1 - \rho) \quad \text{is such that} \quad \pi P = \pi$$

Thus X non-null persistent

Proof of Proposition 38 (2)

Computation for $q < p$: For $i \geq 1$ we have

$$\begin{aligned}
\sum_{j\neq i} p_{ij} y_j &= p_{i,i-1}\, y_{i-1} + p_{i,i+1}\, y_{i+1} \\
&= q\left(1 - \frac{1}{\rho^{i-1}}\right) + p\left(1 - \frac{1}{\rho^{i+1}}\right) \\
&= 1 - \frac{1}{\rho^{i+1}}\left(q\rho^2 + p\right) \\
&= 1 - \frac{1}{\rho^{i+1}}\left(\frac{p^2}{q} + p\right) \\
&= 1 - \frac{p}{\rho^{i+1}}\left(\frac{p}{q} + 1\right) \\
&= 1 - \frac{1}{\rho^{i-1}} \\
&= y_i
\end{aligned}$$

Proof of Proposition 38 (3)

Nbin$(1, 1-\rho)$ distribution: Defined for $k \geq 0$ by

$$\pi_k = \rho^k (1-\rho)$$

Verifying $\pi P = \pi$ for $q > p$: For $j \geq 1$ we have

$$\begin{aligned}
\sum_{i \geq 0} \pi_i p_{ij} &= \pi_{j-1}\, p + \pi_{j+1}\, q \\
&= \rho^{j-1}(1-\rho)p + \rho^{j+1}(1-\rho)q \\
&= \rho^{j-1}(1-\rho)\left(p + \rho^2 q\right) \\
&= \rho^j (1-\rho) \\
&= \pi_j
\end{aligned}$$

Proof of Proposition 38 (4)

Case $q = p$: We have

1. X persistent since
   - $Y \equiv$ random walk is persistent
   - $X = |Y|$

2. X null-persistent since since $x = \mathbf{1}$ is such that

$$x = xP, \quad \text{and} \quad \sum_{i \in S} x_i = \infty$$

Outline

Main objective

Aim in this section:

1. Get expressions for
$$\lim_{n\to\infty} p_{ij}(n)$$
2. Link with stationary distributions

Problem with parity (1)

Example: Take $S = \{1, 2\}$ and

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Question: Can we get

$$\lim_{n \to \infty} p_{ij}(n) \, ?$$

Problem with parity (2)

Example: Take $S = \{1, 2\}$ and

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Behavior of P^n and parity: We find

$$p_{11}(n) = p_{22}(n) = \begin{cases} 0\,, & \text{if } n \text{ is odd} \\ 1\,, & \text{if } n \text{ is even} \end{cases}$$

Thus

$p_{ii}(n)$ does not converge
Problem comes from periodicity

Aperiodic assumption

Hypothesis 39.

Until further notice we assume

X is an irreducible and aperiodic Markov chain

Stationary distributions and return times

Theorem 40.

Let

- X Markov chain with matrix transition P
- X irreducible and aperiodic

Then for all i, j we have

$$\lim_{n\to\infty} p_{ij}(n) = \frac{1}{\mu_j} = \frac{1}{\mathbf{E}[T_j \,|\, X_0 = j]}$$

Some remarks (1)

Persistent null case: If the Markov chain X is persistent null then

$$\lim_{n\to\infty} p_{ij}(n) = 0$$

We had seen this result in Proposition 15

Forgetting the past: If the Markov chain X is non-null persistent then

$$\lim_{n\to\infty} \mathbf{P}\left(X_n = j \,|\, X_0 = i\right) = \lim_{n\to\infty} p_{ij}(n) = 0 = \pi_j = \frac{1}{\mu_j}$$

Thus the initial condition is forgotten

Some remarks (2)

Case with initial distribution: Assume

- X is non-null persistent
- $X_0 \sim \nu$

Then

$$\lim_{n\to\infty} \mathbf{P}\left(X_n = j \,|\, X_0 \sim \mu\right) = \lim_{n\to\infty} \sum_{i\in S} \nu_i p_{ij}(n) = \frac{1}{\mu_j}$$

Example

Definition of the chain: Take $S = \{1, 2, 3\}$ and

$$P = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

Invariant measure: One finds

$$\pi = \begin{pmatrix} .43 & 0 & .57 \end{pmatrix}$$

Large time behavior: One finds (e.g with R)

$$P^{30} = \begin{pmatrix} 43 & 0 & .57 \\ 43 & 9 \times 10^{-10} & .57 \\ 43 & 0 & .57 \end{pmatrix}$$

Periodic case

Theorem 41.

Let

- X Markov chain with matrix transition P
- X irreducible
- X periodic with period d

Then we have

1. $Y = \{Y_n = X_{nd};\, n \geq 0\}$ irreducible aperiodic
2. The following limit holds true:

$$\lim_{n\to\infty} p_{ij}(nd) = \lim_{n\to\infty} \mathbf{P}\left(Y_n = j \,|\, Y_0 = j\right) = \frac{d}{\mu_j}$$

Outline

Reversed chain

Theorem 42.

Let

- X irreducible non-null persistent chain
- Transition for X is P, invariant measure is π
- Hypothesis: $X_n \sim \pi$ for all n
- Set $Y_n = X_{N-n}$ for $0 \leq n \leq N$

Then

1. Y is a Markov chain
2. The transition for Y is

$$\mathbf{P}\left(Y_{n+1} = j \middle| Y_n = i\right) = \frac{\pi_j}{\pi_i} p_{ji}$$

Proof of Theorem 42

Computing conditional probabilities: We have

$$
\begin{aligned}
&\mathbf{P}\left(Y_{n+1} = i_{n+1} \middle| Y_n = i_n, \ldots, Y_0 = i_0\right) \\
&= \frac{\mathbf{P}\left(Y_0 = i_0,\ Y_1 = i_1, \ldots, Y_{n+1} = i_{n+1}\right)}{\mathbf{P}\left(Y_0 = i_0,\ Y_1 = i_1, \ldots, Y_n = i_n\right)} \\
&= \frac{\mathbf{P}\left(X_{N-n-1} = i_{n+1},\ X_{N-n} = i_n, \ldots, X_N = i_0\right)}{\mathbf{P}\left(X_{N-n} = i_n, \ldots, X_N = i_0\right)} \\
&= \frac{\pi_{i_{n+1}}\, p_{i_{n+1} i_n}\, p_{i_n i_{n-1}} \cdots p_{i_1 i_0}}{\pi_{i_n}\, p_{i_n i_{n-1}} \cdots p_{i_1 i_0}} \\
&= \frac{\pi_{i_{n+1}}\, p_{i_{n+1} i_n}}{\pi_{i_n}} \\
&= \mathbf{P}\left(Y_{n+1} = i_{n+1} \middle| Y_n = i_n\right)
\end{aligned}
$$

This gives the Markov property and the transition

Reversed chain

Definition 43.

Let

- X irreducible non-null persistent chain
- Transition for X is P, invariant measure is π
- Hypothesis: $X_n \sim \pi$ for all n
- Set $Y_n = X_{N-n}$ for $0 \leq n \leq N$

Then

1. X is said to be reversible if Y has transition P
2. This is equivalent to

$$\pi_i \, p_{ij} = \pi_j \, p_{ji}, \quad \text{for all} \quad i, j \in S$$

Vocabulary

Detailed balance: Let

- P transition matrix
- λ distribution

Then P, λ are in detailed balance if

$$\lambda_i \, p_{ij} = \lambda_j \, p_{ji}, \quad \text{for all} \quad i, j \in S$$

Reversible in equilibrium: If X is such that

- P, π are in detailed balance,

then X is said to be reversible in equilibrium

Invariant measure and reversibility

Theorem 44.

Let

- X irreducible Markov chain with transition P
- Hypothesis: There exists a distribution π such that

$$\pi_i \, p_{ij} = \pi_j \, p_{ji}, \quad \text{for all} \quad i, j \in S \tag{2}$$

Then

1. π is a stationary distribution
2. X is reversible in equilibrium

Proof of Theorem 44

Computation of πP: We have

$$
\begin{aligned}
(\pi P)_j &= \sum_{i \in S} \pi_i p_{ij} \\
&= \sum_{i \in S} \pi_j p_{ji} \\
&= \pi_j \sum_{i \in S} p_{ji} \\
&= \pi_j
\end{aligned}
$$

Conclusion:

1. π is invariant
2. X is reversible in equilibrium from (2)

Ehrenfest diffusion model (1)

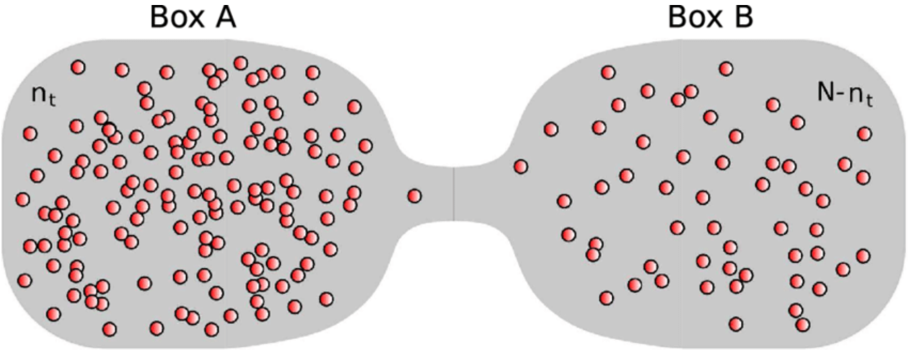
Model: We consider

- Two boxes A and B
- Total of N gas molecules in $A \cup B$
- At time n, one molecule is picked from the N molecules
- This molecule changes box

Process: We set

$$X_n \equiv \# \text{ molecules in Box } A \text{ at time } n$$

Ehrenfest diffusion model (2)



Ehrenfest diffusion model (2)

Proposition 45.

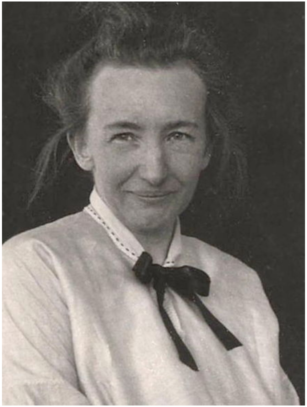
For Ehrenfest's model,

1. X is a Markov chain with
$$p_{i,i+1} = 1 - \frac{i}{m}, \quad \text{and} \quad p_{i,i-1} = \frac{i}{m}$$
2. X is reversible in equilibrium with
$$\pi = \text{Bin}\left(N, \frac{1}{2}\right), \quad \text{that is} \quad \pi_i = \binom{N}{i}\left(\frac{1}{2}\right)^m$$

Tatyana and Paul Ehrenfest

Some facts about the Ehrenfest:

- Lifespan:
  - Tatyana: 1876-1964
  - Paul: 1880-1933
- Born in:
  - Tatyana: Russian empire
  - Paul: Austrian empire
- Contributions in statistical physics
- Problems due to (lack of) religion:
  - Could not marry
  - Difficult to find a job
  - Settled down in Netherlands



Proof of Proposition 45 (1)

Markov chain: One can write

$$X_{n+1} = X_n - (2Y_{n+1} - 1), \quad \text{where} \quad Y_{n+1} \sim \mathcal{B}\left(\frac{X_n}{N}\right)$$

Otherwise stated: We also have

$$X_{n+1} = X_n - \mathbf{1}_{\left(U_{n+1} \leq \frac{X_n}{N}\right)} + \mathbf{1}_{\left(U_{n+1} > \frac{X_n}{N}\right)} \equiv \varphi(X_n, U_{n+1}),$$

where $\{U_k;\, k \geq 1\}$ are i.i.d $\mathcal{U}([0,1])$

Conclusion: X is a Markov chain with

$$p_{i,i+1} = 1 - \frac{i}{m}, \quad \text{and} \quad p_{i,i-1} = \frac{i}{m}$$

Proof of Proposition 45 (2)

Reversible in equilibrium: One checks that

$$\begin{aligned} \pi_i \, p_{i,i+1} &= \pi_{i+1} \, p_{i+1,i} \\ \pi_i \, p_{i,i-1} &= \pi_{i-1} \, p_{i-1,i} \end{aligned}$$

Outline

Irreducible case

Theorem 46.

Let

- X irreducible Markov chain with transition P
- S finite

Then:

X is non-null persistent

Perron-Frobenius theorem

Theorem 47.

Let

- X irreducible Markov chain with transition P
- S finite with $|S| = N$, X has period d

Then:

1. $\lambda_1 = 1$ is an eigenvalue of P
2. Let $\omega = e^{\frac{2\pi\imath}{d}}$. Then the following are eigenvalues of P:

$$\lambda_k = \omega^{k-1}, \quad \text{for} \quad k = 1, \ldots, d$$

3. Remaining eigenvalues:

$$\lambda_{d+1}, \ldots, \lambda_N, \quad \text{with} \quad |\lambda_j| < 1$$

Large time behavior

Theorem 48.

Let

- X irreducible Markov chain with transition P
- S finite with $|S| = N$
- $\Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_N)$ eigenvalue matrix
- Hyp: Eigenvalues λ_j all distinct
- $V = [v_1, \ldots, v_n]$ eigenvector matrix

Then:

1. $P^n = V \Lambda^n V^{-1}$
2. If X is aperiodic we have

$$\lim_{n \to \infty} P^n = V \, \text{Diag}(1, 0, \ldots, 0) V^{-1}$$

Inbreeding model (1)

Model:

- Spinach population
- Genetic information contained in chromosomes
- 6+6 identical pairs of chromosomes
- Sites $C_1, \ldots, C_M$ for chromosomes
  $\hookrightarrow$ We just look at C_1 for 1 chromosome
- $C_1 \in \{a, A\}$ for each pair
- Types: given by $S = \{AA, aA, aa\}$
- $X_n \equiv$ Value of type at generation n for a typical spinach
- Self reproduction model with meiosis
  $\hookrightarrow$ shuffle of C_i's between pairs

Inbreeding model (2)

Transition rules: If all shuffles are equally likely we get

- $AA \times AA \longrightarrow AA$, with $p = 1$
- $aA \times aA \longrightarrow aa$ with $p = \frac{1}{4}$, aA with $p = \frac{1}{2}$, AA with $p = \frac{1}{4}$
- $aa \times aa \longrightarrow aa$, with $p = 1$

Transition matrix: We get

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

Inbreeding model (3)

Classification of states: With the graph we find

- aa and AA are persistent
- aA is transient

Eigenvalues: We find

$$\lambda_1 = 1, \qquad \lambda_2 = 1, \qquad \lambda_3 = \frac{1}{2}$$

Eigenvectors: We get

$$V = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad V^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

Inbreeding model (4)

Large time behavior: We get

$$P^n = V^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \left(\frac{1}{2}\right)^n \end{pmatrix} V$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1} & \left(\frac{1}{2}\right)^n & \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1} \\ 0 & 0 & 1 \end{pmatrix}$$

Limiting behavior: We have

$$\lim_{n\to\infty} P^n = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

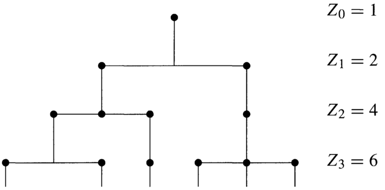
Outline

Assumptions on the model

Main hypotheses:

1. Family sizes are $\perp\!\!\!\perp$ random variables $\{X_i^{(n)};\ i, n \geq 1\}$
2. Family sizes have same pmf f
   $\hookrightarrow$ with generating function G
3. $Z_0 = 1$ and

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_i^{(n+1)}$$



Applying the general theory of Markov chains

What can be said:

1. 0 is an absorbing state, thus persistent non-null
2. All other states are transient
3. Unique invariant measure $\pi = \delta_0$

Partial conclusion:

- This doesn't say much about the behavior of the chain

Combining with generating functions

What more can be said:

1. $\mathbf{P}$(Ultimate extinction) $= \eta$
2. $\eta \equiv$ smallest non-negative root of $s = G(s)$
3. If extinction occurs, then $\lim_{n\to\infty} Z_n = 0$
4. If extinction does not occur, then $\lim_{n\to\infty} Z_n = \infty$

Particular case: If $Z_1 \sim \text{Nbin}(1, p)$ then

$$\eta = \begin{cases} 1 & \text{if } p \leq \frac{1}{2} \\ \frac{q}{p} & \text{if } p > \frac{1}{2} \end{cases}$$

Conditioning

Time of extinction: Define

$$T = \inf\{n;\ Z_n = 0\} \qquad (\text{possibly } T = \infty)$$

and

$$E_n = \{n < T < \infty\}$$

Conditioning: We set

$$\hat{p}_j(n) = \mathbf{P}\left(Z_n = j \middle| E_n\right)$$

Quantity of interest: We wish to compute

$$\hat{\pi}_j \equiv \lim_{n\to\infty} \hat{p}_j(n)$$

Limit of the conditional distribution

Proposition 49.

Consider

- Z branching process
- Hypothesis $\mathbf{E}[Z_1] < \infty$

Then

1. $\hat{\pi}_j \equiv \lim_{n\to\infty} \hat{p}_j(n)$ exists
2. Let $G_{\hat{\pi}} \equiv$ generating function of $\hat{\pi}$. It solves

$$G_{\hat{\pi}}\left(\frac{G(\eta(s))}{\eta}\right) = mG_{\hat{\pi}}(s) + 1 - m, \quad \text{where} \quad m = G'(\eta)$$

Partial proof of Proposition 49 (1)

Definition of $\hat{G}_n$: We set

$$\hat{G}_n(s) = \mathbf{E}\left[s^{Z_n} \middle| E_n \right] = \sum_{j=0}^{\infty} \hat{p}_j(n) s^j$$

More explicit version: We have

$$\begin{aligned} \hat{G}_n(s) &= \sum_{j=0}^{\infty} \frac{\mathbf{P}\left((Z_n = j) \cap E_n\right)}{\mathbf{P}(E_n)} s^j \\ &= \frac{\sum_{j=0}^{\infty} \mathbf{P}\left((Z_n = j) \cap E_n\right) s^j}{\mathbf{P}(E_n)} \end{aligned}$$

Partial proof of Proposition 49 (2)

Expression for $\mathbf{P}((Z_n = j) \cap E_n)$ for $j \geq 1$: We have

$$\mathbf{P}\left((Z_n = j) \cap E_n\right) = \mathbf{P}\left((Z_n = j), \text{ all lines after time } n \text{ die out}\right)$$
$$= \mathbf{P}\left(\text{ all lines after time } n \text{ die out} \,|\, Z_n = j\right) \mathbf{P}\left(Z_n = j\right)$$
$$= \eta^j \mathbf{P}\left(Z_n = j\right)$$

Case $j = 0$: It is easily seen that

$$\mathbf{P}\left((Z_n = 0) \cap E_n\right) = 0$$

Partial proof of Proposition 49 (3)

Partial conclusion: We have obtained

$$\sum_{j=0}^{\infty} \frac{\mathbf{P}\left((Z_n = j) \cap E_n\right)}{\mathbf{P}(E_n)} s^j = G_n(s\eta) - G_n(0)$$

Expression for $\mathbf{P}(E_n)$: Write

$$\mathbf{P}(E_n) = \mathbf{P}(T < \infty) - \mathbf{P}(T \leq n) = \eta - G_n(0)$$

Expression for $\hat{G}_n(s)$: We end up with

$$\hat{G}_n(s) = \frac{G_n(s\eta) - G_n(0)}{\eta - G_n(0)}$$

Partial proof of Proposition 49 (4)

Remainder of the proof: Start from

$$\hat{G}_n(s) = \frac{G_n(s\eta) - G_n(0)}{\eta - G_n(0)}$$

Then

- Use $G_{n+1}(s) = G(G_n(s))$
- Analysis in order to get derivatives from the ratio above

More on the limiting conditional distribution

Proposition 50.

Consider

- Z branching process
- Set $\mu = \mathbf{E}[Z_1]$

Then

1. If $\mu \neq 1$ we have
$$\sum_{j=0}^{\infty} \hat{\pi}_j = 1$$
2. If $\mu = 1$ we have
$$\hat{\pi}_j = 0, \quad \text{for all } j \geq 0$$

Interpretation of Proposition 50

Interpretation:

- If $\mu \neq 1$
  - The distribution $\mathcal{L}(Z_n)$ converges to $\hat{\pi}$
    $\hookrightarrow$ Conditionally on future extinction
- If $\mu = 1$
  - $\lim_{n\to\infty} \mathbf{P}(Z_n = j) = 0$, since extinction is certain
  - $\lim_{n\to\infty} \mathbf{P}(Z_n = j | E_n) = 0$, since $Z_n \to \infty$
    $\hookrightarrow$ Conditionally extinction in the future

Limit in the critical case

Theorem 51.

Consider

- Z branching process
- Hypothesis: $\mu = 1$ and $G''(1) < \infty$
- Set $Y_n = \frac{Z_n}{n\,\sigma^2}$ and $\sigma^2 = \mathbf{Var}(Z_1)$

Then

$$\lim_{n\to\infty} \mathbf{P}(Y_n \leq y | E_n) = 1 - e^{-2y}$$

Interpretation of Theorem 51

Interpretation: Given E_n we have

$$\lim_{n\to\infty} \mathcal{L}(Y_n) = \mathcal{E}(2)$$