Generating functions and their applications

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Introduction to Stochastic Processes – MA 532

Mostly taken from *Probability and Random Processes* by Grimmett-Stirzaker

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Outline

1. Generating functions
2. Random walks
3. Branching processes

Outline

1. Generating functions

2. Random walks

3. Branching processes

Defining generating functions

Definition 1.

Let

- $a = \{a_i;\ i \geq 0\}$ sequence
- $s \in \mathbb{R}$

Then the generating function of a is

$$G_a(s) = \sum_{i=0}^{\infty} a_i s^i \,,$$

provided the series converges

De Moivre's series

Sequence: We consider $\theta \in [0, 2\pi]$ and

$$a_n = e^{\imath n\theta} = [\cos(\theta) + \imath \sin(\theta)]^n$$

Generating function: Defined by

$$G_a(s) = \sum_{n=0}^{\infty} a_n s^n = \sum_{n=0}^{\infty} e^{\imath n\theta} s^n$$

Computation of the generating function: For $|s| < 1$ we get

$$G_a(s) = \frac{1}{1 - s e^{\imath\theta}}$$

Convolution

Definition 2.

Let

- $a = \{a_i;\ i \geq 0\}$ and $b = \{b_i;\ i \geq 0\}$ sequences
- c sequence defined by

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

Then we denote

$$c = a * b$$

Convolution and generating functions

Proposition 3.

Let

- $a = \{a_i;\ i \geq 0\}$ and $b = \{b_i;\ i \geq 0\}$ sequences
- $c = a * b$

Then

$$G_c(s) = G_a(s)\, G_b(s)$$

Proof of Proposition 3

Computation from the definition of G_c: We have

$$\begin{aligned}
G_c(s) &= \sum_{n=0}^{\infty} c_n s^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i} \right) s^n \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n} a_i s^i \, b_{n-i} s^{n-i} \\
&= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} a_i s^i \, b_{n-i} s^{n-i} \\
&= G_a(s) \, G_b(s)
\end{aligned}$$

Poisson random variable (1)

Notation:

$$\mathcal{P}(\lambda) \text{ for } \lambda \in \mathbb{R}_+$$

State space:

$$E = \mathbb{N} \cup \{0\}$$

Pmf:

$$\mathbf{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \geq 0$$

Expected value, variance and pgf:

$$\mathbf{E}[X] = \lambda, \qquad \mathbf{Var}(X) = \lambda, \qquad G_X(s) = \exp(\lambda(s-1))$$

Poisson random variable (2)

Use (examples):

- # customers getting into a shop from 2pm to 5pm
- # buses stopping at a bus stop in a period of 35mn
- # jobs reaching a server from 12am to 6am

Empirical rule:

If $n \to \infty$, $p \to 0$ and $np \to \lambda$, we approximate $\mathrm{Bin}(n, p)$ by $\mathcal{P}(\lambda)$. This is usually applied for

$$p \leq 0.1 \quad \text{and} \quad np \leq 5$$

Poisson random variable (3)

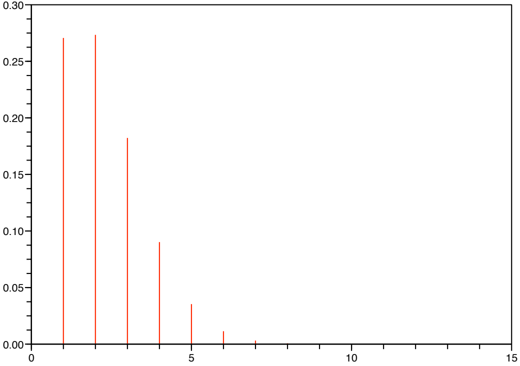


Figure: Pmf of $\mathcal{P}(2)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Poisson random variable (4)

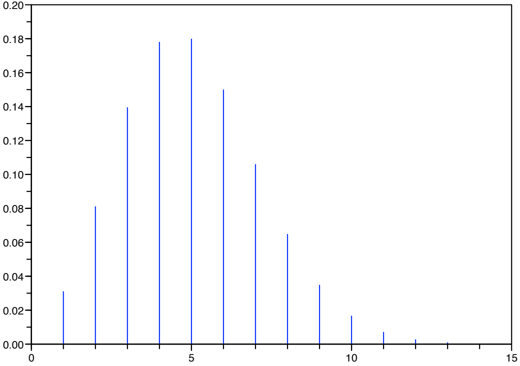
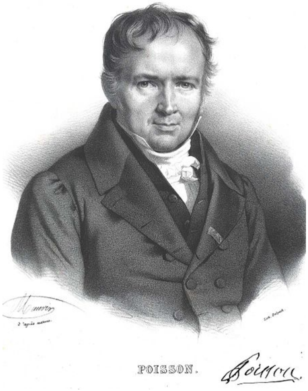


Figure: Pmf of $\mathcal{P}(5)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Siméon Poisson

Some facts about Poisson:

- Lifespan: 1781-1840, in $\simeq$ Paris
- Engineer, Physicist and Mathematician
- Breakthroughs in electromagnetism
- Contributions in partial diff. eq
  celestial mechanics, Fourier series
- Marginal contributions in probability



A quote by Poisson:
Life is good for only two things: doing mathematics and teaching it!!

Sum of 2 Poisson random variables (1)

Question: Consider

- $X \sim \mathcal{P}(\lambda)$, thus $f_X(i) = e^{-\lambda} \frac{\lambda^i}{i!}$
- $Y \sim \mathcal{P}(\mu)$, thus $f_Y(i) = e^{-\mu} \frac{\mu^i}{i!}$
- $X \perp\!\!\!\perp Y$

What is the distribution of $Z = X + Y$?

Sum of 2 Poisson random variables (2)

Pmf for Z: We know that

$$f_Z = f_X * f_Y$$

Generating function for Z: We get

$$\begin{aligned} G_{f_Z}(s) &= G_{f_X}(s)\, G_{f_Y}(s) \\ &= \exp\left((\lambda + \mu)(s - 1)\right) \end{aligned}$$

Conclusion:

$$Z \sim \mathcal{P}(\lambda + \mu)$$

Probability generating functions

Definition 4.

Let

- X random variable with values in $\mathbb{Z}$
- f_X pmf of X

We set

$$G_X(s) = \mathbf{E}\left[s^X\right] = G_{f_X}(s)$$

Properties of the generating function (1)

Some properties:

1. Convergence: There exists $R \geq 0$ such that $G_X(s)$
   - Converges absolutely if $|s| < R$
   - Diverges if $|s| > R$
   - The sum is uniformly convergent on $\{s; \, |s| < R'\}$ if $R' < R$

2. Differentiation: One can differentiate term by term at s
   $\hookrightarrow$ such that $|s| < R$

Properties of the generating function (2)

Some more properties:

3. Uniqueness: Assume
   - $G_a(s) = G_b(s)$ for $|s| < R' \leq R$

   Then
$$(a_n)_{n\geq 0} = (b_n)_{n\geq 0}, \quad \text{and} \quad a_n = \frac{1}{n!} G_a^{(n)}(0)$$

4. Abel theorem: Assume
   - $a_i \geq 0$
   - $G_a(s) < \infty$ for $|s| < 1$

   Then
$$\lim_{s \nearrow 1} G_a(s) = \sum_{i=0}^{\infty} a_i$$

Bernoulli random variable (1)

Notation:

$$X \sim \mathcal{B}(p) \text{ with } p \in (0,1)$$

State space:

$$\{0,1\}$$

Pmf:

$$\mathbf{P}(X=0) = 1-p, \quad \mathbf{P}(X=1) = p$$

Expected value, variance, generating function:

$$\mathbf{E}[X] = p, \qquad \mathbf{Var}(X) = p(1-p), \qquad G_X(s) = (1-p) + p\,s$$

Bernoulli random variable (2)

Use 1, success in a binary game:

- Example 1: coin tossing
  - $X = 1$ if H, $X = 0$ if T
  - We get $X \sim \mathcal{B}(1/2)$
- Example 2: dice rolling
  - $X = 1$ if outcome $= 3$, $X = 0$ otherwise
  - We get $X \sim \mathcal{B}(1/6)$

Use 2, answer yes/no in a poll

- $X = 1$ if a person feels optimistic about the future
- $X = 0$ otherwise
- We get $X \sim \mathcal{B}(p)$, with unknown p

Jacob Bernoulli

Some facts about Bernoulli:

- Lifespan: 1654-1705, in Switzerland
- Discovers constant e
- Establishes divergence of $\sum \frac{1}{n}$
- Contributions in diff. eq
- First law of large numbers
- Bernoulli:
  family of 8 prominent mathematicians
- Fierce math fights between brothers

Geometric random variable

Notation:

$$X \sim \mathcal{G}(p), \qquad \text{for } p \in (0,1)$$

State space:

$$E = \mathbb{N} = \{1, 2, 3, \dots\}$$

Pmf:

$$\mathbf{P}(X = k) = p\,(1-p)^{k-1}, \quad k \geq 1$$

Expected value, variance and generating function:

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \mathbf{Var}(X) = \frac{1-p}{p^2}, \qquad G_X(s) = \frac{p\,s}{1 - s(1-p)}$$

Geometric random variable (2)

Use:

- Independent trials, with $\mathbf{P}(\text{success}) = p$
- $X = \#$ trials until first success

Example: dice rolling

- Set $X =$ 1st roll for which outcome $= 6$
- We have $X \sim \mathcal{G}(1/6)$

Computing some probabilities for the example:

$$\mathbf{P}(X = 5) = \left(\frac{5}{6}\right)^4 \frac{1}{6} \simeq 0.08$$

$$\mathbf{P}(X \geq 7) = \left(\frac{5}{6}\right)^6 \simeq 0.33$$

Geometric random variable (3)

Computation of $\mathbf{E}[X]$: Set $q = 1 - p$. Then

$$\begin{aligned}
\mathbf{E}[X] &= \sum_{i=1}^{\infty} i q^{i-1} p \\
&= \sum_{i=1}^{\infty} (i-1) q^{i-1} p + \sum_{i=1}^{\infty} q^{i-1} p \\
&= q\,\mathbf{E}[X] + 1
\end{aligned}$$

Conclusion:

$$\mathbf{E}[X] = \frac{1}{p}$$

Generating function and moments

Theorem 5.

Let X be a random variable with generating function G_X. Then

1. $\mathbf{E}[X] = G'(1)$
2. $\mathbf{E}[X(X-1)\cdots(X-k+1)] = G_X^{(k)}(1)$

Remark: If the radius of convergence for G_X is 1, then

$$G_X^{(k)}(1) = \lim_{s \nearrow 1} G_X^{(k)}(s)$$

Computing moments with generating functions

Situation: Consider $p \in (0,1)$ and

$$X \sim \mathcal{G}(p)$$

Derivatives of G_X: We find

$$\begin{aligned} G_X'(s) &= \frac{p}{(1-(1-p)s)^2} \\ G_X''(s) &= \frac{2p(1-p)}{(1-(1-p)s)^3} \end{aligned}$$

Moments: We get

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \mathbf{Var}(X) = \frac{1-p}{p^2}$$

Generating function for a sum

Theorem 6.

Let

- X, Y random variables
- $X \perp\!\!\!\perp Y$

Then

$$G_{X+Y}(s) = G_X(s)G_Y(s)$$

Binomial random variable (1)

Notation:

$$X \sim \text{Bin}(n, p), \text{ for } n \geq 1, \; p \in (0, 1)$$

State space:

$$\{0, 1, \ldots, n\}$$

Pmf:

$$\mathbf{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad 0 \leq k \leq n$$

Expected value, variance and generating function:

$$\mathbf{E}[X] = np, \qquad \mathbf{Var}(X) = np(1 - p), \qquad G_X(s) = [(1 - p) + p\, s]^n$$

Binomial random variable (2)

Use 1, Number of successes in a Bernoulli trial:

- Example: Roll a dice 9 times.
- $X = \#$ of 3 obtained
- We get $X \sim \text{Bin}(9, 1/6)$
- $\mathbf{P}(X = 2) = 0.28$

Use 2: Counting a feature in a repeated trial:

- Example: stock of 1000 pants with 10% defects
- Draw 15 times a pant at random
- $X = \#$ of pants with a defect
- We get $X \sim \text{Bin}(15, 1/10)$

Binomial random variable (3)

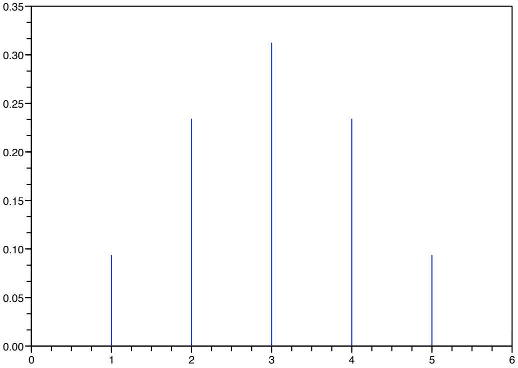


Figure: Pmf for $\mathrm{Bin}(6; 0.5)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Binomial random variable (4)

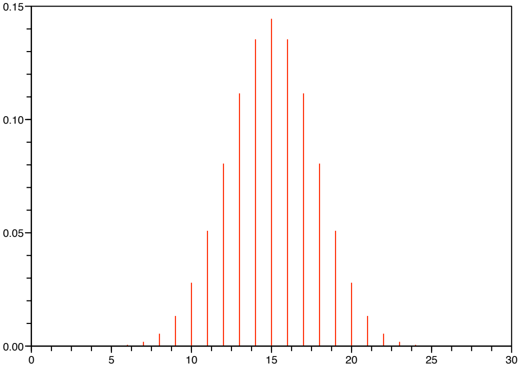


Figure: Pmf for $\text{Bin}(30; 0.5)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Computation for G_X

Generating function for Bernoulli: If $Y \sim \mathcal{B}(p)$ then

$$G_Y(s) = (1-p) + p\,s$$

Decomposition of Binomial: If $X \sim \text{Bin}(n,p)$ one can write

$$X = \sum_{i=1}^{n} Y_i, \quad \text{with} \quad Y_i \text{ i.i.d, } Y_i \sim \mathcal{B}(p)$$

Computing G_X: We get

$$G_X(s) = \prod_{i=1}^{n} G_{Y_i}(s) = [(1-p) + p\,s]^n$$

Joint generating functions

Definition 7.

Let

- X_1, X_2 random variables
- X_1, X_2 take values in $\mathbb{Z}$

Then the pgf for (X_1, X_2) is

$$G_{X_1,X_2}(s_1, s_2) = \mathbf{E}\left[s_1^{X_1} s_2^{X_2}\right]$$

Characterization of independence

Theorem 8.

Let

- X_1, X_2 random variables
- G_{X_1,X_2} the corresponding pgf

Then we have

$$X_1 \perp\!\!\!\perp X_2 \quad \Longleftrightarrow \quad G_{X_1,X_2}(s_1, s_2) = G_{X_1}(s_1) G_{X_2}(s_2) \text{ for all } s_1, s_2$$

Outline

Definition of random walk

Definition 9.

Let

- $X_1, \ldots, X_n$ Bernoulli random variables with values ± 1,

$$\mathbf{P}(X_i = 1) = p, \qquad \mathbf{P}(X_i = -1) = 1 - p$$

- The X_i's are independent

We set $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i$$

Then S is called simple random walk

Symmetric random walk

Definition 10.

Let

- $X_1, \ldots, X_n$ Bernoulli random variables with values ± 1,

$$\mathbf{P}(X_i = 1) = \frac{1}{2}, \qquad \mathbf{P}(X_i = -1) = \frac{1}{2}$$

- The X_i's are independent

We set $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i$$

Then S is called symmetric random walk

Illustration: 30 steps of a random walk

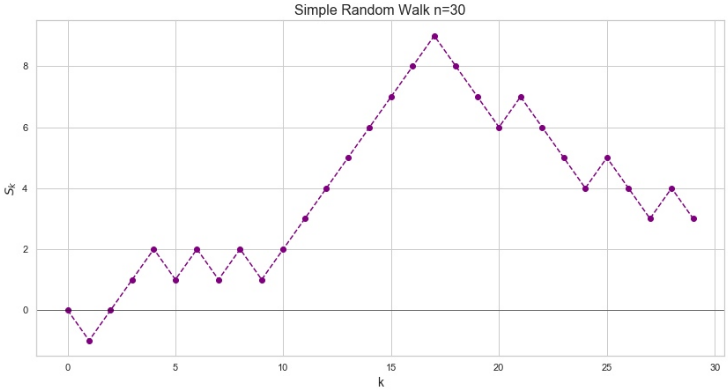


Illustration: chaotic path (Brownian motion)

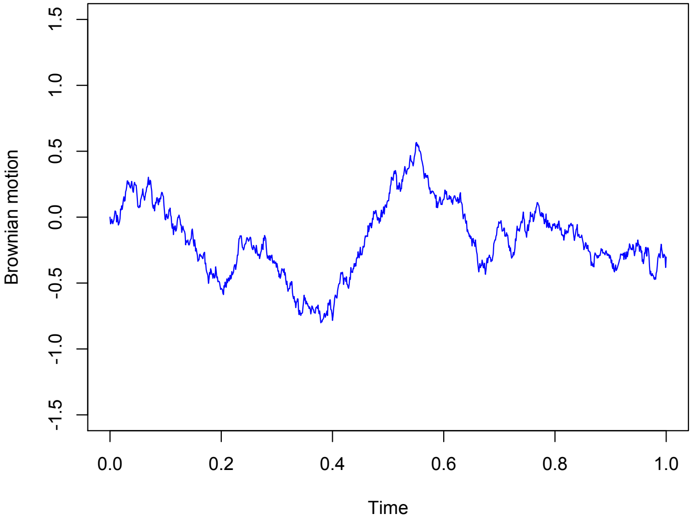
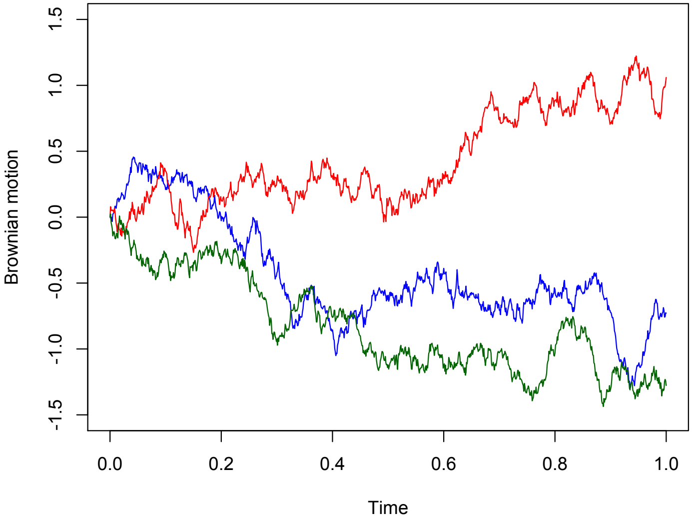


Illustration: random path (Brownian motion)



Questions about random walks

Main questions

1. Does the walk S_n go to ∞ when $n \to \infty$?
2. Does it return to 0 after $n = 0$?
3. How often does it return to 0?
4. What is the range of $S_n(\omega)$?

Methodologies to answer those questions

1. Elementary methods based on generating functions
2. Later: Markov chain methods
3. Also useful: martingale methods

Notation (1)

Return time to 0: We set $T_0 = \infty$ if there is no return to 0, and

$$T_0 = \inf \{n > 0;\ S_n = 0\}$$

Probability to be at origin after n steps: We set

$$p_0(n) = \mathbf{P}(S_n = 0)$$

Probability that 1st return occurs after n steps: Define

$$f_0(n) = \mathbf{P}(T_0 = n) = \mathbf{P}(S_1 \neq 0, \ldots, S_{n-1} \neq 0, S_n = 0)$$

Notation (2)

Generating functions: We set

$$P_0(s) = \sum_{n=0}^{\infty} p_0(n) s^n, \qquad F_0(s) = \sum_{n=1}^{\infty} f_0(n) s^n$$

Probabilistic interpretation: We have

$$F_0(s) = \mathbf{E}\left[s^{T_0}\right]$$

Warning: T_0 is a defective random variable. Thus we have

- $s^{T_0} = 0$ if $T_0 = \infty$ if $s \in [0, 1)$
- This is also valid as $s \nearrow 1$ (hence "$1^\infty = 0$" here)
- Thus $F_0(1) = \mathbf{P}(T_0 < \infty)$

Computing P_0 and F_0

Theorem 11.

Let S_n be the random walk with parameters p and $q = 1 - p$. Then

1. P_0 and F_0 satisfy
$$P_0(s) = 1 + P_0(s)F_0(s)$$
2. P_0 verifies
$$P_0(s) = \frac{1}{(1 - 4pqs^2)^{1/2}}$$
3. F_0 is given by
$$F_0(s) = 1 - \left(1 - 4pqs^2\right)^{1/2}$$

Proof of Theorem 11 (1)

Events: We set

$$A = (S_n = 0), \qquad B_k = (T_0 = k)$$

Decomposition for A: We have

$$A = A \cap \left(\bigcup_{k=1}^{n} B_k \right) = \bigcup_{k=1}^{n} (A \cap B_k)$$

Decomposition for $\mathbf{P}(A)$: We get

$$\mathbf{P}(A) = \sum_{k=1}^{n} \mathbf{P}(A \cap B_k) = \sum_{k=1}^{n} \mathbf{P}(A | B_k)\, \mathbf{P}(B_k) \tag{1}$$

Proof of Theorem 11 (2)

Convolution relation: Equation (1) can be read as

$$p_0(n) = \sum_{k=1}^{n} p_0(n-k) f_0(k), \quad \text{for} \quad n \geq 1, \quad \text{and} \quad p_0(0) = 1$$

Expression with generating functions: We get

$$P_0(s) = 1 + P_0(s) F_0(s)$$

Proof of Theorem 11 (3)

Computing $p_0(n)$: We have

1. For n odd,
$$p_0(n) = 0$$
2. For n even, one argue that
    - $(S_n = 0) \iff$ equal # steps up and steps down
    - There is $\binom{n}{n/2}$ ways to choose the up steps
    - Probability of each sequence leading to 0: $(p\,q)^{n/2}$

Thus for n even we have

$$p_0(n) = \mathbf{P}(S_n = 0) = \binom{n}{n/2}(p\,q)^{n/2}$$

Proof of Theorem 11 (4)

First expression for P_0: We have found

$$P_0(n) = \sum_{m=0}^{\infty} \binom{2m}{m} (p\,q)^m s^{2m} = \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2} (p\,q\,s^2)^m$$

A binomial series: We have

$$\frac{1}{(1+x)^{1/2}} = \sum_{m=0}^{\infty} \frac{(-1)^m (2m)!}{4^m (m!)^2} x^m = \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2} \left(-\frac{x}{4}\right)^m$$

Second expression for P_0: We get

$$P_0(s) = \frac{1}{(1-4pqs^2)^{1/2}}$$

Proof of Theorem 11 (5)

Summary: We have obtained

$$
\begin{aligned}
P_0(s) &= 1 + P_0(s)F_0(s) \\
P_0(s) &= \frac{1}{(1-4pqs^2)^{1/2}}
\end{aligned}
$$

Conclusion: We easily get

$$F_0(s) = 1 - \left(1 - 4pqs^2\right)^{1/2}$$

Information on T_0

Proposition 12.

Recall that

$$T_0 = \inf \{n > 0;\ S_n = 0\}$$

Then

1. We have
$$\mathbf{P}(T_0 < \infty) = 1 - |p - q|$$
2. In particular $T_0 < \infty$ almost surely iff $p = q = \frac{1}{2}$
3. If $p = \frac{1}{2}$, then
$$\mathbf{E}[T_0] = \infty$$

Proof of Proposition 12 (1)

Expression for F_0: We have seen

$$F_0(s) = 1 - \left(1 - 4pqs^2\right)^{1/2}$$

Expression for $\mathbf{P}(T_0 < \infty)$: We have also seen that

$$\mathbf{P}(T_0 < \infty) = F_0(1)$$

Hence

$$\begin{aligned} \mathbf{P}(T_0 < \infty) &= F_0(1) \\ &= 1 - (1 - 4pq)^{1/2} \\ &= 1 - |2p - 1| \\ &= 1 - |p - q| \end{aligned}$$

Proof of Proposition 12 (2)

F_0 for $p = 1/2$: When $p = q = \frac{1}{2}$ we have

$$F_0(s) = 1 - \left(1 - s^2\right)^{1/2}$$

Expression for $\mathbf{E}[T_0]$: We have seen that

$$\mathbf{E}[T_0] = F_0'(1)$$

Computation of F_0': We get

$$F_0'(s) = \frac{s}{(1 - s^2)^{1/2}}$$

Conclusion: We have

$$\mathbf{E}[T_0] = F_0'(1) = \infty$$

Vocabulary

Persistent or recurrent: the random walk is said to be recurrent
$\hookrightarrow$ iff $\mathbf{P}(T_0 < \infty) = 1$

Transient: the random walk is said to be transient
$\hookrightarrow$ iff $\mathbf{P}(T_0 < \infty) < 1$

Summarizing our result: We have seen that

$$\text{Random walk is persistent} \iff p = \frac{1}{2}$$

Visits to point r

Definition 13.

The first time to visit r is defined by

$$T_r = \inf\{n > 0;\ S_n = r\}$$

Then we set

$$f_r(n) = \mathbf{P}(T_r = n) = \mathbf{P}(S_1 \neq r, \ldots, S_{n-1} \neq r, S_n = r)$$

and

$$F_r(s) = \sum_{n=1}^{\infty} f_r(n) s^n$$

Generating function for T_r

Theorem 14.

For $r \geq 1$ we have

$$F_r(s) = [F_1(s)]^r$$

with

$$F_1(s) = \frac{1 - (1 - 4pqs^2)^{1/2}}{2q}$$

Proof of Theorem 14 (1)

Events: We set, for $r > 1$,

$$A = (T_r = n), \qquad B_k = (T_{r-1} = n - k)$$

Decomposition for A: We have

$$A = A \cap \left(\bigcup_{k=1}^{n-1} B_k \right) = \bigcup_{k=1}^{n-1} (A \cap B_k)$$

Decomposition for $\mathbf{P}(A)$: We get

$$\mathbf{P}(A) = \sum_{k=1}^{n-1} \mathbf{P}(A \cap B_k) = \sum_{k=1}^{n-1} \mathbf{P}(A | B_k) \, \mathbf{P}(B_k) \tag{2}$$

Proof of Theorem 14 (2)

Convolution relation: Equation (2) can be read as

$$f_r(n) = \sum_{k=1}^{n-1} f_1(k) f_{r-1}(n-k), \quad \text{for} \quad n \geq 1, \quad \text{and} \quad f_r(0) = 0$$

Expression with generating functions: We get

$$F_r(s) = F_1(s) F_{r-1}(s)$$

Conclusion for F_r: Iterating the above relation we get

$$F_r(s) = [F_1(s)]^r$$

Proof of Theorem 14 (3)

Conditioning on X_1: For $n > 1$ we have

$$\begin{aligned} \mathbf{P}(T_1 = n) &= \mathbf{P}(T_1 = n \mid X_1 = 1)p + \mathbf{P}(T_1 = n \mid X_1 = -1)q \\ &= 0 + \mathbf{P}\left(\text{1st visit to 1 takes } n-1 \text{ steps} \mid S_0 = -1\right) q \\ &= \mathbf{P}(T_2 = n-1)q \end{aligned}$$

Relation on pmf's: We get, for $n > 1$

$$f_1(n) = q f_2(n-1)$$

Relation on generating functions: Multiplying by s^n we obtain

$$\begin{aligned} F_1(s) &= ps + sqF_2(s) \\ &= ps + sq\left(F_1(s)\right)^2 \end{aligned}$$

Proof of Theorem 14 (4)

Recall: We have obtained

$$F_1(s) = ps + sq\,(F_1(s))^2$$

Expression for F_1:
Solving for $F_1(s)$ in the quadratic equation we get

$$F_1(s) = \frac{1 - (1 - 4pqs^2)^{1/2}}{2q}$$

Visits to the upper half plane

Proposition 15.

Let S_n be the random walk
$\hookrightarrow$ with parameters p and $q = 1 - p$.

Then

$$\mathbf{P}(\text{At least one visit to the upper half plane}) = \min\left(1, \frac{p}{q}\right)$$

Proof of Proposition 16

Notation: Set

$$A = \text{At least one visit to the upper half plane}$$

Expression with generating function: We have

$$\begin{aligned} \mathbf{P}(A) &= \mathbf{P}(T_1 < \infty) \\ &= F_1(1) \\ &= \frac{1 - |p - q|}{2q} \end{aligned}$$

Conclusion: Separating cases $p > q$ and $p \leq q$ we get

$$\mathbf{P}(A) = \min\left(1, \frac{p}{q}\right)$$

Hitting time theorem

Theorem 16.

Let

- S_n be the random walk with parameters p and $q = 1 - p$
- $b \in \mathbb{Z}^*$ and $n \geq 1$
- $T_b = \inf \{n > 0;\ S_n = b\}$

Then

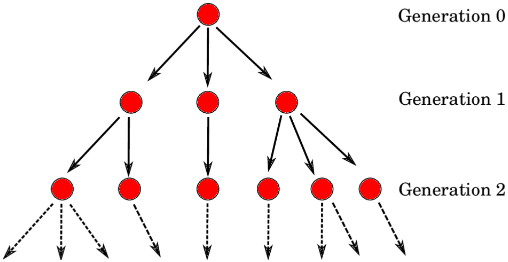
$$\mathbf{P}(T_b = n) = \frac{|b|}{n} \mathbf{P}(S_b = n)$$

Outline

Model

Model for population evolution:

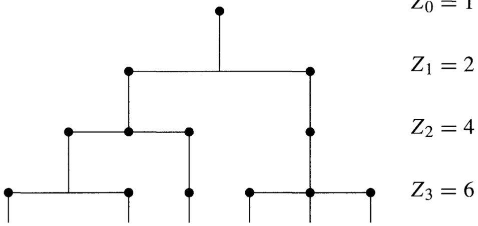
- $Z_n \equiv \#$ individuals of n-th generation
- At n-th generation: each member gives birth
  $\hookrightarrow$ To a $\#$ individuals of $(n+1)$-th generation
- Family size: random variable



Assumptions on the model

Main hypotheses:

1. Family sizes form collection of $\perp\!\!\!\perp$ random variables
2. Family sizes have same pmf f
   $\hookrightarrow$ with generating function G
3. $Z_0 = 1$



Generating functions for random sums

Theorem 17.

Let

- $\{X_j;\, j \geq 1\}$ sequence of i.i.d random variables
- $G_X \equiv$ common generating function
- N random variable, with $N \perp\!\!\!\perp (X_j)_{j\geq 1}$ and $N \in \{0, 1, \ldots\}$
- $G_N \equiv$ generating function for N
- $Z = \sum_{j=1}^{N} X_j$

Then

$$G_Z(s) = G_N(G_X(s))$$

Proof of Theorem 17

Computation: We have

$$\begin{aligned} G_Z(s) &= \mathbf{E}[s^Z] \\ &= \sum_{n=0}^{\infty} \mathbf{E}\left[s^Z \,|\, N = n\right] \mathbf{P}(N = n) \\ &= \sum_{n=0}^{\infty} \mathbf{E}\left[s^{\sum_{j=1}^{N} X_j} \,|\, N = n\right] \mathbf{P}(N = n) \\ &= \sum_{n=0}^{\infty} \mathbf{E}\left[s^{\sum_{j=1}^{n} X_j}\right] \mathbf{P}(N = n) \\ &= \sum_{n=0}^{\infty} (G_X(s))^n f_N(n) \\ &= G_N(G_X(s)) \end{aligned}$$

Generating function for the branching process

Theorem 18.

For the branching process, recall that

- $Z_n = \#$ individuals of n-th generation
- $G =$ generating function for the offspring f

We set

$$G_n(s) = \mathbf{E}\left[s^{Z_n}\right]$$

Then

$$G_{m+n}(s) = G_m(G_n(s)) = G_n(G_m(s))$$

Thus

$$G_n(s) = G^{\circ(n)}(s)$$

Proof of Theorem 18 (1)

Decomposition of Z_{n+m}: Write

$$\begin{aligned} Z_{n+m} &= Y_1 + \cdots + Y_{Z_m} \\ &= \sum_{j=1}^{Z_m} Y_j, \end{aligned}$$

where

$$Y_j = \# \text{ individuals in generation } (n+m) \text{ which stem from individual } j \text{ in } m\text{-th generation}$$

Proof of Theorem 18 (2)

Recall:

$$Z_{n+m} = \sum_{j=1}^{Z_m} Y_j$$

Information on the random variables Y_j:

- Y_j's are independent
- Y_j's are independent of Z_m
- $Y_j \overset{(d)}{=} Z_n$

Application of Theorem 17:

$$G_{m+n}(s) = G_m(G_{Y_1}(s)) = G_m(G_n(s))$$

Moments of Z_n

Proposition 19.

For the branching process with offspring $Z_1 \sim f$ set

$$\mu = \mathbf{E}[Z_1], \qquad \sigma^2 = \mathbf{Var}(Z_1)$$

Then

$$\mathbf{E}[Z_n] = \mu^n$$

and

$$\mathbf{Var}(Z_n) = \begin{cases} n\,\sigma^2 & \text{if } \mu = 1 \\ \frac{\sigma^2(\mu^n - 1)\mu^{n-1}}{\mu - 1} & \text{if } \mu \neq 1 \end{cases}$$

Proof of Proposition 19 (1)

Method of computation: We use

$$\mathbf{E}[Z_n] = G_n'(1)$$

Recursive relation: Recall that

$$G_n(s) = G(G_{n-1}(s))$$

Proof of Proposition 19 (2)

Recall: We have

$$G_n(s) = G(G_{n-1}(s))$$

Differentiate: We have

$$G_n'(s) = G'\left(G_{n-1}(s)\right) G_{n-1}'(s)$$

Thus at $s = 1$ we get

$$\mathbf{E}[Z_n] = G'(1)\,\mathbf{E}[Z_{n-1}] = \mu\,\mathbf{E}[Z_{n-1}]$$

Conclusion: Since $\mathbf{E}[Z_0] = 1$, we get

$$\mathbf{E}[Z_n] = \mu^n$$

Proof of Proposition 19 (3)

Method for the variance: We use

$$\mathbf{E}[Z_n(Z_n-1)] = G_n''(1)$$

or

$$\mathbf{Var}(Z_n) = G_n''(1) + G_n'(1) - \left(G_n'(1)\right)^2$$

Recursive relation: We differentiate twice the relation

$$G_n(s) = G(G_{n-1}(s))$$

We get a linear recursion (to be solved)

$$\begin{aligned} G_n''(1) &= G''(1)\left(G_{n-1}'(1)\right)^2 + G'(1)G_{n-1}''(1) \\ &= \left(\sigma^2 + \mu(\mu-1)\right)\mu^{2(n-1)} + \mu G_{n-1}''(1) \end{aligned}$$

Negative binomial random variable (1)

Notation:

$$X \sim \text{Nbin}(r, p), \text{ for } r \in \mathbb{N}^*, \ p \in (0,1)$$

State space:

$$\{0, 1, 2 \ldots\}$$

Pmf:

$$\mathbf{P}(X = k) = \binom{k+r-1}{k} p^r \, q^k, \quad k \geq 0$$

Expected value, variance and pgf:

$$\mathbf{E}[X] = \frac{r\,q}{p}, \qquad \mathbf{Var}(X) = \frac{r\,q}{p^2}, \qquad G_X(s) = \left(\frac{p}{1-(1-p)s}\right)^r$$

Negative binomial random variable (2)

Use:

- Independent trials, with $\mathbf{P}(\text{success}) = p$
- $X = \#$ failures until r successes

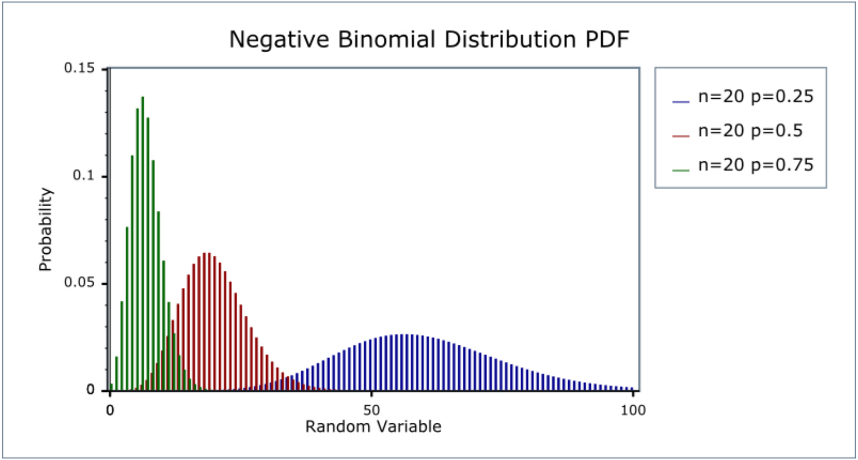
Justification:

$$(X = k) \\ = \\ ((r-1) \text{ successes in } (k+r-1) \text{ 1st trials}) \\ \cap\ ((k+r)\text{-th trial is a success})$$

Thus

$$\mathbf{P}(X = k) = \binom{k-1}{r-1} p^r q^k$$

Negative binomial random variable (3)



Negative binomial random variable for $r = 1$

Notation:

$$X \sim \text{Nbin}(1, p), \text{ for } r \in \mathbb{N}^*, \; p \in (0, 1)$$

State space:

$$\{0, 1, 2 \ldots\}$$

Pmf:

$$\mathbf{P}(X = k) = p\, q^k, \quad k \geq 0$$

Expected value, variance and pgf:

$$\mathbf{E}[X] = \frac{q}{p}, \qquad \mathbf{Var}(X) = \frac{q}{p^2}, \qquad G_X(s) = \frac{p}{1 - (1 - p)s}$$

Branching with negative binomial offspring

Proposition 20.

For the branching process with $Z_1 \sim \text{Nbin}(1, p)$ we have

1. The generating function G_n is given by

$$G_n(s) = \begin{cases} \frac{n-(n-1)s}{n+1-ns} & \text{if } p = \frac{1}{2} \\ \frac{q(p^n-q^n)-ps\left(p^{n-1}-q^{n-1}\right)}{p^{n+1}-q^{n+1}-ps(p^n-q^n)} & \text{if } p \neq \frac{1}{2} \end{cases}$$

2. The probability of extinction is

$$\mathbf{P}(\text{Ultimate extinction}) = \begin{cases} 1 & \text{if } p \leq \frac{1}{2} \\ \frac{q}{p} & \text{if } p > \frac{1}{2} \end{cases}$$

Proof of Proposition 20 (1)

Pgf for Z_1: Since $Z_1 \sim \text{Nbin}(1, p)$ we have

$$G_{Z_1}(s) = \frac{p}{1 - (1 - p)s}$$

Expression for G_n: One can check that

$$G\left(G_n(s)\right) = G_{n+1}(s)$$

Proof of Proposition 20 (2)

Ultimate extinction: We set

$$A = (\text{Ultimate extinction occurs})$$

Then

$$A = \bigcup_{n \geq 1} A_n, \quad \text{with} \quad A_n = (Z_n = 0)$$

$\mathbf{P}(A)$ as a limit: We have

$$A_n \subset A_{n+1} \quad \Longrightarrow \quad \mathbf{P}(A) = \lim_{n \to \infty} \mathbf{P}(A_n)$$

Proof of Proposition 20 (3)

Expression for $\mathbf{P}(A_n)$: We have

$$\mathbf{P}(A_n) = G_n(0) = \begin{cases} \frac{n}{n+1} & \text{if } p = \frac{1}{2} \\ \frac{q(p^n - q^n)}{p^{n+1} - q^{n+1}} & \text{if } p \neq \frac{1}{2} \end{cases}$$

Expression for $\mathbf{P}(A)$: We obtain

$$\mathbf{P}(A) = \lim_{n \to \infty} \mathbf{P}(A_n) = \begin{cases} 1 & \text{if } p \leq \frac{1}{2} \\ \frac{q}{p} & \text{if } p > \frac{1}{2} \end{cases}$$

Ultimate extinction in the general case

Theorem 21.

Consider a branching process with

- $Z_1 \sim f$, f with pgf G
- $\mu = \mathbf{E}[Z_1]$ and $\sigma^2 = \mathbf{Var}(Z_1)$

Let

- $\eta \equiv$ smallest non-negative root of $s = G(s)$

Then

1. $\mathbf{P}(\text{Ultimate extinction}) = \eta$
2. $\eta = 1$ if $\mu < 1$
3. $\eta < 1$ if $\mu > 1$
4. $\eta = 1$ if $\mu = 1$ and $\sigma^2 > 0$

Galton-Watson process

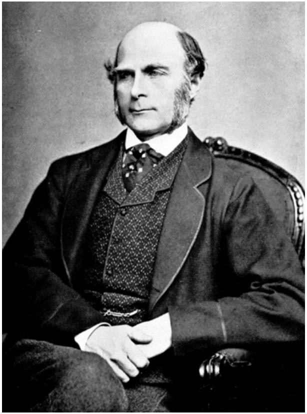
Historical facts:

- Francis Galton proposed Theorem 21 as a problem in 1869
- Galton was interested in survival of family names
- Problem solved by Watson in 1874
- Watson's solution used a method still presented today
- $\{Z_n;\ n \geq 0\}$ is often referred to as Galton-Watson process

Francis Galton: the bright side

Some facts about Galton:

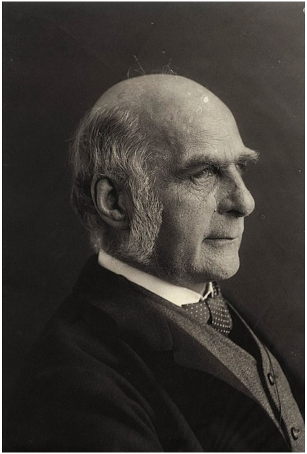
- Lifespan: 1822-1911, in England
- Polymath
- First use of stats in surveys
- Founded psychometry
- Founded meteorology
- Invented Galton whistle
- Was Darwin's cousin



Francis Galton: the dark side

Uneasy facts about Galton:

- Founded eugenics
  $\hookrightarrow$ Twist on Darwin's theory
- Coined the term eugenics
- "Nature vs nurture"
- Very controversial views on race
- UCL removed his name in 2020
  $\hookrightarrow$ From a large lecture room



Proof of Theorem 21 (1)

Ultimate extinction: Recall that we have set

$$A = (\text{Ultimate extinction occurs})$$

Then

$$A = \bigcup_{n \geq 1} A_n, \quad \text{with} \quad A_n = (Z_n = 0)$$

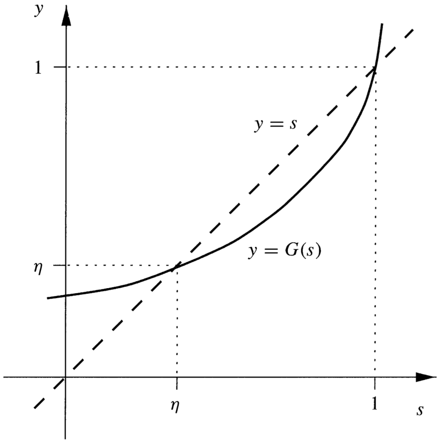
$\mathbf{P}(A)$ as a limit: We have $A_n \subset A_{n+1}$. Thus

$$\eta_n \equiv \mathbf{P}(A_n) \text{ is } \nearrow, \quad \text{and} \quad \mathbf{P}(A) = \lim_{n\to\infty} \eta_n$$

Proof of Theorem 21 (2)

Claim when $\mu > 1$:

$$G(0) \in [0,1)\ ,\ G'(0) \in [0,1)\ ,\ G'(1) > 1\ ,\ G \text{ convex on } [0,1]$$



Proof of Theorem 21 (3)

Claim $G(0) \in [0,1)$: We have

$$G(0) = \mathbf{P}(Z_1 = 0) < 1 \quad \text{(otherwise trivial extinction)}$$

Claim $G'(0) \in [0,1)$: Write

$$G'(0) = \mathbf{P}(Z_1 = 1) < 1 \quad \text{(or trivial offspring} = 1)$$

Claim $G'(1) > 1$: One argues

$$G'(1) = \mu > 1$$

Claim G convex on $[0,1]$: We compute

$$G''(s) = \mathbf{E}\left[Z_1(Z_1 - 1)s^{Z_1 - 2}\right] \geq 0$$

Proof of Theorem 21 (4)

Conclusion: Follows classical lines for sequences

$$\eta_{n+1} = G(\eta_n) \quad \Longrightarrow \quad \lim_{n\to\infty} \eta_n = \eta$$