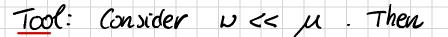


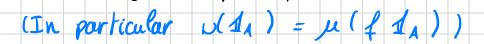
existence + uniqueness of E[XIF]? -> proved



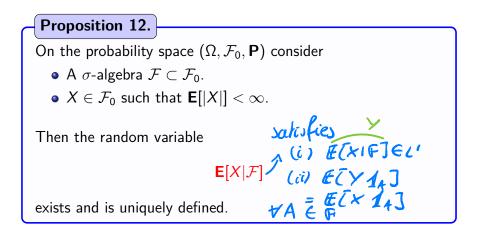
## ∃ f ≥O, measurable such

## that for all $a \in B_b$

 $\nu(g) = \mu(fg)$ 



#### Conditional expectation: existence



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### (1) Refine a measure 1 on (S,F) by setting

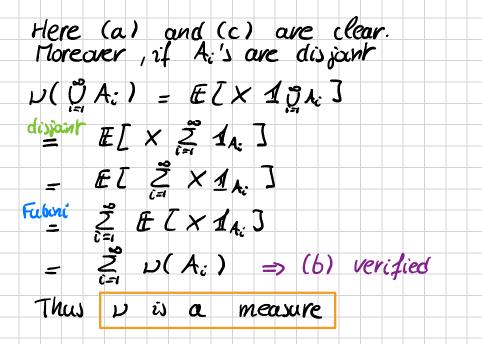
## $\mathcal{V}(A) = \mathbb{E}[X \mathbf{1}_{A}], \forall A \in \mathcal{F}$

Claim: N is a measure, i.e

# (a) $U(\phi) = 0$ , disjoint union

(b)  $\cup (\bigcup_{i=1}^{n} A_i^{-}) = \sum_{i=1}^{n} \cup (A_i^{-})$ 

#### (c) $U(A) \ge 0 + A \in F$

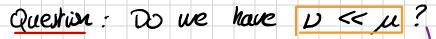


## $\frac{\text{Recall}}{\text{Recall}} : \mathcal{N}(A) = E[X \ \mathbf{1}_{A}]$

Take  $\mu = \mathbb{P} \cdot \mu$  is a probability

on (l, Fo), but also on (l, F)

## We can write $\mu(A) = E \overline{L} 1_A \overline{J}$

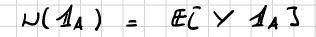


If  $\mu(AI=0)$ , then  $I_A = 0$  a.s.

 $\Rightarrow$  X  $1_A = 0 a$ . Yes!

 $\Rightarrow \mathcal{U}(A) = E[X 1_A] = 0$ 

#### Radon - Nykodym: Since U << U 3 YEF St. ¥ A E F



## Thus YEF and for all AEF

## $E[X 1_A] = E[Y 1_A]$

Conclusion: Y= E[×16]

#### Proof of existence

#### Hypothesis: We have

- A  $\sigma$ -algebra  $\mathcal{F} \subset \mathcal{F}_0$ .
- $X \in \mathcal{F}_0$  such that  $\mathbf{E}[|X|] < \infty$ .
- $X \ge 0$ .

#### Defining two measures: we set

• 
$$\mu = P$$
, measure on  $(\Omega, \mathcal{F})$ .

Then  $\nu$  is a measure (owing to Beppo-Levi).

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#### Proof of existence (2)

Absolute continuity: we have

$$\mathbf{P}(A) = 0 \Rightarrow \mathbf{1}_A = 0 \quad P\text{-a.s.}$$
$$\Rightarrow X \mathbf{1}_A = 0 \quad P\text{-a.s.}$$
$$\Rightarrow \nu(A) = 0$$

Thus  $\nu \ll P$ 

Conclusion: invoking Radon-Nykodym, there exists  $f \in \mathcal{F}$  such that, for all  $A \in \mathcal{F}$ , we have  $\nu(A) = \int_A f \, d\mathbf{P}$ .  $\hookrightarrow$  We set  $f = \mathbf{E}[X|\mathcal{F}]$ .

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#### Outline

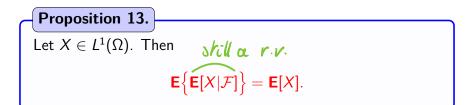
#### Definition

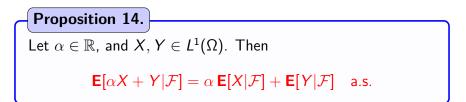
- Baby conditional distributions: discrete case
- Baby conditional distributions: continuous case
- Definition with measure theory

#### 2 Examples

- 3 Existence and uniqueness
- 4 Conditional expectation: properties
- 5 Conditional expectation as a projection
- 6 Conditional regular laws

#### Linearity, expectation





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### Proof of lineanty. Set

#### Z= XE[XIF]+ E[YIF]

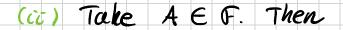
#### We want to prove (i) and (ii) for z

## (c) ZEF, since

## E(XIF)EF, E[XIF]EF

and z is a linear combination of the two.

## Z= & E[XIF]+ E[YIF]



 $E[ \neq 1_{A}] rv. rv.$ =  $E\{(\alpha E[X|F], E[Y|F]) 1_{A}\}$ 

# E linear & E { E [ × 1 G] 1 A } + E { E [ × 1 G] 1 A }

# $\stackrel{(ii)}{=} \propto E[\times 1_A] + E[\times 1_A]$

linearity E ( (x+ Y) IA]

=> (ii) verified

Proof that ELE[XIF] = E[X]

#### We have that 44 EF,

### $E[Y 1_A] = E[X 1_A]$

## In particular, REF. We get

## ElY12] = ElX12]

## $\Rightarrow E[Y] = E[X]$

#### Proof

Strategy: Check (i) and (ii) in the definition for the r.v

$$Z \equiv \alpha \, \mathbf{E}[X|\mathcal{F}] + \mathbf{E}[Y|\mathcal{F}].$$

Verification: we have

(i) Z is a linear combination of  $\mathbf{E}[X|\mathcal{F}]$  and  $\mathbf{E}[Y|\mathcal{F}]$  $\hookrightarrow Z \in \mathcal{F}$ .

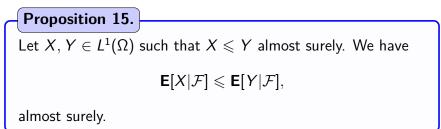
(ii) For all  $A \in \mathcal{F}$ , we have

$$\mathbf{E}[Z \mathbf{1}_{A}] = E\left\{ \left( \alpha \mathbf{E}[X|\mathcal{F}] + \mathbf{E}[Y|\mathcal{F}] \right) \mathbf{1}_{A} \right\} \\ = \alpha E\left\{ \mathbf{E}[X|\mathcal{F}] \mathbf{1}_{A} \right\} + E\left\{ \mathbf{E}[Y|\mathcal{F}] \mathbf{1}_{A} \right\} \\ = \alpha \mathbf{E}[X \mathbf{1}_{A}] + \mathbf{E}[Y \mathbf{1}_{A}] \\ = \mathbf{E}[(\alpha X + Y) \mathbf{1}_{A}].$$

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#### Monotonicity

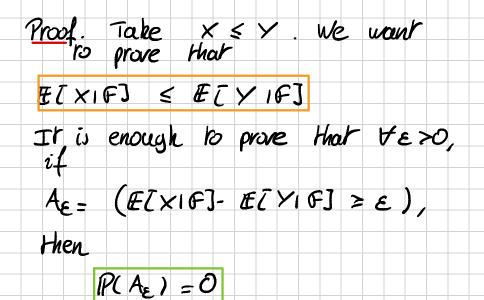


**Proof**: Along the same lines as proof of uniqueness for the conditional expectation. For instance if we set

$$A_{\varepsilon} = \{\mathbf{E}[X|\mathcal{F}] - \mathbf{E}[Y|\mathcal{F}] \ge \varepsilon > 0\},\$$

then it is readily checked that

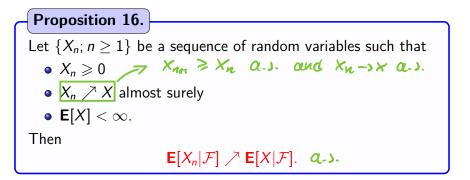
$$\mathbf{P}(A_{\varepsilon}) = 0.$$



#### $A_{\varepsilon} = (E[X|F] - E[Y|F] \ge \varepsilon) \in F$

≥E on he We have EP(AE) & E {(E[XIF] - E[YIF]) 1AE Linearity of E[.15] E{ E[(X-Y) IF] 1/E ] ( TEL (X-Y) 1AE] 5 O Thus E P(AE) *≦0*  $\Rightarrow P(A_2) = 0$ 

#### Monotone convergence



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Recall: Xn 7 Xa.s. We set

## Yn= X-Xn. We have Yn 20 a.s.

#### Set Zn=E[YnIF]. We have

#### (i) Since E[. IF] is monstane and Yn >, we have

# $Z_n \supseteq (Z_{n+1} \leq Z_n \quad a.s.)$

## (ii) Yn 20 => E[Yn 16] 20

#### Thus JZo ZO J.r. Zn > Zo



 $Y_n \equiv X - X_n$   $Z_n \equiv E[Y_n (F]$   $Z_n \gg Z_\infty \quad a.s.$   $Z_\infty \ge 0 \quad a.s.$ 

we wish to prove : 20 = 0

#### Rmk: If Zo ≥0, in order to prove that Zo =0, it is enough to prove E[Zo]=0

However  $E[z_n] = E\{E[X_n|G]\}$ =  $E[Y_n]$ 

In addition, Ya > O => E[Ya] > O

We have obtained

EZZn] > O

In addition,

Thus

 $z_n \ge z_\infty \quad \alpha.$ Bego-Levi => E[Zn] >> E[Zo]

 $E[z_{\infty}] = 0 = 2 = 2 = 0 \alpha.$ 

=> Beppo-Levi for E[-15]

#### Proof

Strategy: Set  $Y_n \equiv X - X_n$ . We are reduced to show  $Z_n \equiv \mathbf{E}[Y_n | \mathcal{F}] \searrow 0$ .

Existence of a limit:  $n \mapsto Y_n$  is decreasing, and  $Y_n \ge 0$  $\hookrightarrow Z_n$  is decreasing and  $Z_n \ge 0$ .  $\hookrightarrow Z_n$  admits a limit a.s, denoted by  $Z_{\infty}$ .

Aim: Show that  $Z_{\infty} = 0$ .

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#### Proof (2)

Expectation of  $Z_{\infty}$ : we will show that  $\mathbf{E}[Z_{\infty}] = 0$ . Indeed

- $X_n$  converges a.s. to X.
- $0 \leq X_n \leq X \in L^1(\Omega).$

Thus, by dominated convergence,  $\mathbf{E}[X_n] \rightarrow \mathbf{E}[X]$ .

We deduce:

- $\mathbf{E}[Y_n] \to 0$
- Since  $\mathbf{E}[Y_n] = \mathbf{E}[Z_n]$ , we also have  $\mathbf{E}[Z_n] \to 0$ .

• By monotone convergence, we have  $\mathbf{E}[Z_n] \to \mathbf{E}[Z_\infty]$ This yields  $\mathbf{E}[Z_\infty] = 0$ .

Conclusion:  $Z_{\infty} \ge 0$  and  $\mathbf{E}[Z_{\infty}] = 0$  $\hookrightarrow Z_{\infty} = 0$  almost surely.