**Def** $\{X_n; n \geq 0\}$ is a martingale if

(i) $X_n \in \mathcal{F}_n$ (ii) $X_n \in L^1(\Omega)$

(iii) $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n$

**Example** $X_n = \sum_{i=1}^{n} Z_i$ (simple rw)

is a martingale if $\mathbb{P}(Z_i = \pm 1) = \frac{1}{2}$

it is Markov for any iid $Z_i$'s

with $\begin{cases} \mathbb{P}(Z_i = 1) = p \\ \mathbb{P}(Z_i = -1) = 1-p \end{cases}$, $\forall p \in (0,1)$

# Conditional expectation in the past

**Proposition 3.**

Let $X$ be a $\mathcal{F}_n$-martingale and $m \geq 0$.
For all $n \geq m$ we have

$$\mathbf{E}[X_n | \mathcal{F}_m] = X_m.$$

Proof: Recursive procedure.

Important corollary: Let $X$ be a $\mathcal{F}_n$-martingale and $m \geq 0$.
For all $n \geq m$ we have

$$\mathbf{E}[X_n] = \mathbf{E}[X_m] = \mathbf{E}[X_0]. \tag{1}$$

Proof of prop 3 . By induction

(i) $n = m$ . Then

$$\mathbb{E}[X_n \mid \mathcal{F}_m] = \mathbb{E}[\overset{\in \mathcal{F}_m}{X_m} \mid \mathcal{F}_m] = X_m$$

(ii) Assume $\mathbb{E}[X_n \mid \mathcal{F}_m] = X_m$. Then

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_m] \overset{\mathcal{F}_m \subset \mathcal{F}_n}{=} \mathbb{E}\{\underbrace{\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]}_{= X_n} \mid \mathcal{F}_m\}$$

$$= \mathbb{E}[X_n \mid \mathcal{F}_m] \overset{\text{Hyp}}{=} X_m$$

Induction works!

Consequence of Prop 3: $\forall n \geq m$,

$$\mathbb{E}[X_n] = \mathbb{E}[X_m] = \cdots = \mathbb{E}[X_0]$$

Proof

$$\mathbb{E}[X_n] = \mathbb{E}\{ \overbrace{\mathbb{E}[X_n \mid \mathcal{F}_m]}^{= X_m \text{ from Prop 3}} \}$$

$$= \mathbb{E}[X_m]$$

# Composition with a convex function

**Proposition 4.**

Let

- $X$ a $\mathcal{F}_n$-martingale.
- $\varphi : \mathbb{R} \to \mathbb{R}$ a convex function
  $\hookrightarrow$ such that $\varphi(X_n) \in L^1(\Omega)$ for all $n \geq 0$.
- $Y_n = \varphi(X_n)$

Then $Y$ is a submartingale.

Proof: application of Jensen for conditional expectation.

Example: If $X_n$ is a random walk, $X_n^2$ is a submartingale
$\hookrightarrow$ Fluctuations increase with time.

Proof Set $Y_n = \varphi(X_n)$. We have

(i) $X_n \in \mathcal{F}_n$, $\varphi$ is measurable

$\Rightarrow \varphi(X_n) \in \mathcal{F}_n$

(ii) By assumption, $\varphi(X_n) = Y_n \in L^1$

(iii) We want to prove

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] \geq Y_n$$

# Computation

convex

$$\mathbb{E}[\,Y_{n+1}\mid\mathcal{F}_n\,] = \mathbb{E}[\,\varphi(X_{n+1})\mid\mathcal{F}_n\,]$$

$$\overset{\text{Jensen}}{\geq} \varphi\big(\overbrace{\mathbb{E}[\,X_{n+1}\mid\mathcal{F}_n\,]}^{=X_n}\big)$$

$$= \varphi(X_n)$$

$$= Y_n$$

$$\Rightarrow \boxed{\mathbb{E}[\,Y_{n+1}\mid\mathcal{F}_n\,] \geq Y_n}$$

# Outline

# Martingale transformation

**Definition 5.**

Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration and $X, H$ $\mathcal{F}_n$-adapted-processes.

- We say that $H$ is predictable if $H_n \in \mathcal{F}_{n-1}$.
- The transform of $X$ by $H$ is

$$[H \cdot X]_n = \sum_{j=1}^{n} H_j \, \Delta X_j, \quad \text{where} \quad \Delta X_j = X_j - X_{j-1}$$

Interpretation:

1. $H \equiv$ game strategy
   $\hookrightarrow$ Today's decision depends on the information until yesterday
2. $H \cdot X \equiv$ value if strategy $H$ is used

**Def** $[H\cdot X]_n = \sum_{i=1}^{n} \overset{\in \mathcal{F}_{j-1}}{H_j} \Delta X_j \qquad \Delta X_j = X_j - X_{j-1}$

**Interpretation:** $[H\cdot X]_n$ describes the evolution of your wealth when you follow the strategy $H$.

Call $G_n$ your fortune at time $n$. You divide $G_n$ into $H_n$ units of an asset with value $X_n$.

$$G_n = H_n X_n + (G_n - H_n X_n)$$

$$\Rightarrow \quad G_{n+1} = H_n X_{n+1} + (G_n - H_n X_n)$$

$$\Rightarrow \quad G_{n+1} - G_n = H_n \Delta X_{n+1}$$

# Origin of the word martingale:

strategy for a game allowing to win almost surely

D'Alembert's martingale is one of the most famous examples

# D'Alembert

Some facts about d'Alembert:

- Abandoned after birth
- Mathematician
- Contribution in fluid dynamics
- Philosopher
- Participation in 1st Encyclopedia

$\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$

**Game:** Flip a coin

$$\begin{cases} \xi_i = 1 & \text{if } i\text{-th flip is } H \\ \phantom{\xi_i =} -1 & \text{otherwise} \end{cases}$$

$\Rightarrow \sum_{i=1}^{n} \xi_i = X_n$ is a rw, thus a martingale

**Strategy:** $H_1 = 1 \in \mathcal{F}_0$

$H_2 = 2$ if $\xi_1 = -1$, otherwise $H_2 = 0$

$\vdots$

$H_j = 2^{j-1} \mathbb{1}(\xi_{j-1} = -1) \in \mathcal{F}_{j-1}$

$\mathbb{P}(\xi_i = \pm 1) = \frac{1}{2}$

Set $N = \inf\{n \geq 1;\ \xi_n = 1\}$

Claim: $N < \infty$ a.s. In fact $N \sim \mathcal{G}(\frac{1}{2})$

Claim: This strategy allows to win $\$1$ a.s. We have

$$[H \cdot X]_n = \sum_{j=1}^{n} H_j \, \Delta X_j$$

$$\Rightarrow \boxed{[H \cdot X]_N} = \sum_{j=1}^{N} H_j \, \Delta X_j = \sum_{j=1}^{N-1} 2^{j-1}(-1) + 2^{N-1}(1)$$

$$= 2^{N-1} - \sum_{j=1}^{N-1} 2^{j-1} = 2^{N-1} - \frac{2^{N-1}-1}{2-1}$$

$$\boxed{= 1}$$

**Problem** : In order to win \$1, we have to bet

$$Z = 2^N - 1 \quad , \quad N \sim \mathcal{G}\left(\tfrac{1}{2}\right)$$

$$\Rightarrow \quad \boxed{\mathbb{E}[Z]} = \sum_{n=1}^{\infty} 2^n \, \mathbb{P}(N=n) - 1$$

$$= \sum_{n=1}^{\infty} 2^n \times \frac{1}{2^n} - 1$$

$$\boxed{= \infty}$$

# D'Alembert's Martingale

Example: Let $X_n = \sum_{i=1}^n \xi_i$ be a random walk.
We interpret $\xi_i$ as a gain ou a loss at $i$th iteration of the game.
The filtration is $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$

Strategy: We define $H$ in the following way:

- $H_1 = 1$, thus $H_1 \in \mathcal{F}_0$.
- $H_n = 2\, H_{n-1}\, \mathbf{1}_{(\xi_{n-1}=-1)}$

Let $N = \inf\{j \geq 1;\ \xi_j = 1\}$. Then

$$[H \cdot X]_N = \sum_{j=1}^{N} H_j\, \Delta X_j = \sum_{j=1}^{N} H_j\, \xi_j = -\sum_{j=1}^{N-1} 2^{j-1} + 2^{N-1} = 1$$

We get an almost sure gain!

# Strategies and martingales

**Theorem 6.**

Let

- $X$ a martingale.
- $H$ a predictable process such that $H_j \Delta X_j \in L^1$ for all $j$.

Then $H \cdot X$ is a martingale.

Interpretation: One cannot win in a fair game context
$\hookrightarrow$ Compare with d'Alembert's martingale

Proof $\quad [H \cdot X]_n = \sum_{j=1}^{n} H_j \, \Delta X_j$

(i) $\quad [H \cdot X]_n = \sum_{j=1}^{n} \underbrace{H_j}_{\in \mathcal{F}_{j-1} \subset \mathcal{F}_n} \, \Delta X_j) \in \mathcal{F}_j \subset \mathcal{F}_n$

$\Rightarrow \quad [H \cdot X]_n \in \mathcal{F}_n$

(ii) $\quad$ Hyp: $\quad H_j \, \Delta X_j \in \mathcal{L}^1 \quad \forall j$

$\Rightarrow \quad [H \cdot X]_n = \sum_{j=1}^{n} H_j \, \Delta X_j \in \mathcal{L}^1$

$$[H\cdot X]_n = \sum_{j=1}^{n} H_j \,\Delta X_j$$

(iii) $$[H\cdot X]_{n+1} = [H\cdot X]_n + H_{n+1}\Delta X_{n+1}$$

$$\Rightarrow \mathbb{E}\left[ [H\cdot X]_{n+1} \mid \mathcal{F}_n \right]$$

linearity

$$= \mathbb{E}\left[ \overbrace{[H\cdot X]_n}^{\in \mathcal{F}_n} \mid \mathcal{F}_n \right] + \mathbb{E}\left[ \overbrace{H_{n+1}}^{\in \mathcal{F}_n} \Delta X_{n+1} \mid \mathcal{F}_n \right]$$

$$= [H\cdot X]_n + H_{n+1} \overbrace{\mathbb{E}\left[ X_{n+1} - X_n \mid \mathcal{F}_n \right]}^{=0}$$

(martingale prop)

$$= [H\cdot X]_n$$

# Proof

Main ingredients: We write

$$[H \cdot X]_{n+1} = [H \cdot X]_n + H_{n+1} \left( X_{n+1} - X_n \right).$$

Then we use the fact that

1. $H$ is predictable
2. $X$ is a martingale

# Stopping time

Interpretation: If we observe $X_0, \dots, X_n$, we know if $(T \le n)$ or $(T > n)$

**Definition 7.**

Let $= \bar{\mathbb{N}} = \{0, 1, \dots\} \cup \{\infty\}$

- $T : \Omega \to \bar{N}$ random time
- $\mathcal{F}_n$ a filtration

We say that $T$ is a stopping time for $\mathcal{F}_n$ if

- For all $n \in \mathbb{N}$, the set $\{\omega;\ T(\omega) = n\}$ is $\mathcal{F}_n$-measurable.

Recall: basic examples are hitting times.

$N = \inf\{j \ge 1;\ \xi_j = 1\} \Rightarrow$

$(N = n) = (\xi_1 = -1) \cap \cdots \cap (\xi_{n-1} = -1) \cap (\xi_n = 1) \in \mathcal{F}_n$

In general, if $X_n$ is a martingale and

any $A \in \mathcal{B}(\mathbb{R})$

$$T = \inf\{ j \geq 0 ; \ X_j \in A \}$$

is a stopping time:

$$(T = n) = (X_0 \notin A) \cap \cdots \cap (X_{n-1} \notin A) \cap (X_n \in A) \in \mathcal{F}_n$$

Spoiler: $X_n$ martingale and $X_n \geq 0$ a.s.

$\Rightarrow$ $X_n \xrightarrow{a.s.} X_\infty$ for $X_\infty \in L^1$