Outline

Generalized Markov's inequality

Proposition 1.

Let

- $h : \mathbb{R} \to [0, \infty)$ non-negative function
- X random variable with $h(X) \in L^1(\Omega)$

Then for all $a > 0$ we have

$$\mathbf{P}\left(h(X) \geq a\right) \leq \frac{\mathbf{E}\left[h(X)\right]}{a}$$

measurable

$h(x) \geq 0 \quad \forall x$

Proof Define

$$A = \{ \omega ; \; h(X(\omega)) \geq a \} \quad (\in \mathcal{F})$$

Then
$$h(X) = \overbrace{h(X) \mathbb{1}_A}^{\geq a} + \overbrace{h(X) \mathbb{1}_{A^c}}^{\geq 0}$$

$$\Rightarrow h(X) \geq a \, \mathbb{1}_A$$

$h(X) \in L^1$
$$\Rightarrow \mathbb{E}[h(X)] \geq a \, \mathbb{P}(A)$$

$$\Rightarrow \boxed{\mathbb{P}(h(X) \geq a) \leq \frac{\mathbb{E}[h(X)]}{a}}$$

$$\mathbb{P}(h(X) \geq a) \leq \frac{\mathbb{E}[h(X)]}{a}$$

Case 1 : Take $h(x) = |x|$. Then

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}[|X|]}{a} \quad \text{(Markov)}$$

Case 2: Take $h(x) = x^2$. Then

$$\boxed{\mathbb{P}(|X| \geq a)} = \mathbb{P}(X^2 \geq a^2)$$

$$\boxed{\leq \frac{\mathbb{E}[X^2]}{a^2}}$$

Generalizations

- $\mathbb{P}(|X| \geq a) \leq \dfrac{\mathbb{E}[|X|^n]}{a^n}$

- $\underbrace{\mathbb{P}(|X| \geq a) \leq \mathbb{E}[e^{\beta x}]\, e^{-\beta a}}$

  ↳ Large deviations

Proof of Proposition 1

Deterministic inequality: Set

$$A = \{h(X) \geq a\}$$

Then we have

$$h(X) \geq a\,\mathbf{1}_A$$

Expectations: Taking expectations above, we get

$$\mathbf{E}\left[h(X)\right] \geq a\,\mathbf{P}(A)$$

Particular cases of Proposition 1:

Case $h(X) = |X|$: We get Markov's inequality,

$$\mathbf{P}(|X| \geq a) \leq \frac{\mathbf{E}[|X|]}{a}$$

Case $h(X) = X^2$: We get Chebyshev's inequality,

$$\mathbf{P}(|X| \geq a) \leq \frac{\mathbf{E}[|X|^2]}{a^2}$$

Reversed Markov type inequality

Proposition 2.

Let

- $h : \mathbb{R} \to [0, M)$ non-negative bounded function
- X random variable

Then for all $0 < a < M$ we have

$$\mathbf{P}\left(h(X) \geq a\right) \leq \frac{\mathbf{E}\left[h(X)\right] - a}{M - a}$$

$\geq$

Proof Consider $A = \{h(X) \geq a\}$. Then

$$h(X) = \overbrace{h(X)}^{\leq M} \mathbb{1}_A + \overbrace{h(X)}^{\leq a} \mathbb{1}_{A^c} \qquad h: \mathbb{R} \to [0, M)$$

$$h(X) \leq M \, \mathbb{1}_A + a \, \mathbb{1}_{A^c}$$

$$\overset{\mathbb{E}}{\Rightarrow} \quad \mathbb{E}[h(X)] \leq M \, \mathbb{P}(A) + a(1 - \mathbb{P}(A))$$

$$\Rightarrow \quad \mathbb{E}[h(X)] \leq (M - a)\, \mathbb{P}(A) + a$$

$$\Rightarrow \quad \boxed{\mathbb{P}(A) \geq \frac{\mathbb{E}[h(X)] - a}{M - a}}$$

Proof of Proposition 2

Deterministic inequality: Set

$$A = \{h(X) \geq a\}$$

Then we have

$$h(X) \leq M \, \mathbf{1}_A + a \, \mathbf{1}_{A^c}$$

Expectations: Taking expectations above, we get

$$\mathbf{E}[h(X)] \leq M \, \mathbf{P}(A) + a \, (1 - \mathbf{P}(A))$$
$$\Longrightarrow \mathbf{P}(A) \leq \frac{\mathbf{E}[h(X)] - a}{M - a}$$

Hölder's inequality

Proposition 3.

Let

- X, Y random variables
- $p, q > 1$ such that $p^{-1} + q^{-1} = 1$

Then we have

$$\|X\,Y\|_{L^1} \leq \|X\|_{L^p} \|Y\|_{L^q}$$

Minkowski's inequality

Proposition 4.

Let

- X, Y random variables
- $p \geq 1$

Then we have

$$\|X + Y\|_{L^p} \leq \|X\|_{L^p} + \|Y\|_{L^p}$$

Remark:
$(L^p(\Omega), \|\cdot\|_{L^p})$ is a Banach space

Limits of sums

Proposition 5.

Let

- X, Y random variables
- X_n, Y_n sequences of random variables

Then we have

1. $X_n \xrightarrow{\text{a.s}} X$ and $Y_n \xrightarrow{\text{a.s}} Y \implies X_n + Y_n \xrightarrow{\text{a.s}} X + Y$
2. $X_n \xrightarrow{L^p} X$ and $Y_n \xrightarrow{L^p} Y \implies X_n + Y_n \xrightarrow{L^p} X + Y$
3. $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y \implies X_n + Y_n \xrightarrow{P} X + Y$
4. $X_n \xrightarrow{(d)} X$ and $Y_n \xrightarrow{(d)} Y \not\Rightarrow X_n + Y_n \xrightarrow{(d)} X + Y$

Example of $X_n \xrightarrow{(d)} X$, $Y_n \xrightarrow{(d)} Y$, but

$$X_n + Y_n \not\xrightarrow{(d)} X + Y$$

- Take $X_n = Z \sim \mathcal{U}([0,1])$
  $Y_n = W = 1 - Z \sim \mathcal{U}([0,1])$
- Thus $X_n \xrightarrow{(d)} \mathcal{U}([0,1])$ $\quad Y_n \xrightarrow{(d)} \mathcal{U}([0,1])$
- However $X_n + Y_n \overset{a.s.}{=} 1 \sim \delta_1$
- Thus $\boxed{X_n + Y_n \not\xrightarrow{(d)} \text{"}\mathcal{U}([0,1]) + \mathcal{U}([0,1])\text{"}}$

Counter-example for Proposition 5 - item 4

Example of sequence: Consider

- $X_n = X \sim \mathcal{B}(\frac{1}{2})$
- $Y_n = Y = 1 - X$

Convergences: We have

$$X_n \xrightarrow{(d)} X, \qquad Y_n \xrightarrow{(d)} X\,.$$

However

$$X_n + Y_n \xrightarrow{(d)} 1$$

Outline

Limsup of sets

Definition 6.

Let

- $\{A_n;\, n \geq 1\}$ sequence in $\mathcal{F}$

We define

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

Interpretation: We also have

$$\limsup_{n\to\infty} A_n = \{\omega \in \Omega;\, \omega \text{ belongs to an infinity of } A_n\text{'s}\}$$

Borel-Cantelli lemma

Q. What if $\sum_n \mathbf{P}(A_n) = \infty$?
Do we have $\mathbf{P}(\limsup A_n) > 0$?

Theorem 7.

Let

- $\{A_n;\, n \geq 1\}$ sequence in $\mathcal{F}$

We assume

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) < \infty$$

Then we have

$$\mathbf{P}\left(\limsup_{n\to\infty} A_n\right) = 0$$

Reversed Borel-Cantelli lemma

Theorem 8.

Conclusion: If the A_n's are ⫫, then $P(\limsup A_n) \in \{0,1\}$

Let

- $\{A_n;\, n \geq 1\}$ sequence in $\mathcal{F}$

We assume

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty, \quad \text{and} \quad A_n\text{'s independent}$$

Then we have

$$\mathbf{P}\left(\limsup_{n\to\infty} A_n\right) = 1$$