Stochastic differential equations

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Probability Theory 2 - MA 539

Outline

Outline

Aim

Coefficients: We consider

- $\alpha \in \mathbb{R}^n$ and $b, \sigma^1, \ldots, \sigma^d : \mathbb{R}^n \to \mathbb{R}^n$.
- We denote: $\sigma = (\sigma^1, \ldots, \sigma^d) : \mathbb{R}^n \to \mathbb{R}^{n \times d}$
- W, d-dimensional Brownian motion.

Equation: We wish to solve

$$dX_s = b(X_s)\, ds + \sum_{j=1}^{d} \sigma^j(X_s)\, dW_s^j.$$

Integral form: With Itô's integral,

$$X_t = \alpha + \int_0^t b(X_s)\, ds + \sum_{j=1}^{d} \int_0^t \sigma^j(X_s)\, dW_s^j. \tag{1}$$

Infinitesimal drift and covariance

Proposition 1.

Let b, σ bounded, X solution of (1). Then:

$$\partial_t \mathbf{E}\left[X_t \middle| \mathcal{F}_s\right]_{|_{t=s}} = b(X_s), \quad \partial_t \mathbf{Cov}\left(X_t \middle| \mathcal{F}_s\right)_{|_{t=s}} = a(X_s),$$

with $a = \sigma \, \sigma^* : \mathbb{R}^n \to \mathbb{R}^{n \times n}$.

Interpretation:

- $b(X_s) \equiv$ Infinitesimal drift.
- $a(X_s) \equiv$ Infinitesimal covariance.

Itô process

Definition 2.

Let:

- $X : [0, \tau] \to \mathbb{R}^n$ process in L_a^2
- $\alpha \in \mathbb{R}^n$ initial condition.
- b bounded and adapted process in $\mathbb{R}^n$.
- $\{\sigma^k;\ k = 1, \ldots d\}$ process of $L_a^2([0, \tau]; \mathbb{R}^n)$.

We say that X is an Itô process if it admits a decomposition:

$$X_t = \alpha + \int_0^t b_s\, ds + \sum_{k=1}^{d} \int_0^t \sigma_s^k\, dW_s^k.$$

Remark:

A solution of (1) is an Itô process

$\hookrightarrow$ With $b_s = b(X_s)$ and $\sigma_s = \sigma(X_s)$.

Itô's formula for a solution of (1)

Proposition 3.

Let:

- X Itô process, defined by α, b, σ.
- $f \in \mathcal{C}_b^2$.

Then $f(X_t)$ can be decomposed as:

$$\begin{aligned} f(X_t) &= f(\alpha) + \sum_{j=1}^{n} \int_0^t \partial_{x_j} f(X_r)\, b_r^j\, dr \\ &\quad + \sum_{j=1}^{n} \sum_{k=1}^{d} \int_0^t \partial_{x_j} f(X_r)\, \sigma_r^{jk}\, dW_r^k \\ &\quad + \frac{1}{2} \sum_{j_1, j_2=1}^{n} \sum_{k=1}^{d} \int_0^t \partial^2_{x_{j_1} x_{j_2}} f(X_r)\, \sigma_r^{j_1 k}\, \sigma_r^{j_2 k}\, dr. \end{aligned}$$

Proof of Proposition 1

Simplification: This will be shown for $s = 0$
$\hookrightarrow$ No conditional expectation to consider.

Drift term: Start from equation

$$X_t = \alpha + \int_0^t b(X_s)\, ds + \sum_{j=1}^{d} \int_0^t \sigma^j(X_s)\, dW_s^j.$$

The terms $\int_0^t \sigma^j(X_s)\, dW_s^j$ are centered. Therefore:

$$\mathbf{E}[X_t] = \alpha + \int_0^t \mathbf{E}[b(X_s)]\, ds.$$

and

$$\partial_t \mathbf{E}[X_t]_{|_{t=0}} = \mathbf{E}[b(X_t)]_{|_{t=0}} = b(\alpha).$$

Proof of Proposition 1 (2)

Coordinate product: Let

- $l, m \in \{1, \ldots, n\}$.
- $f : \mathbb{R}^n \to \mathbb{R}$ defined by the product $f(x) = x^l x^m$.
- $a_s = \sigma_s \sigma_s^*$.

According to Itô's formula we have:

$$
\begin{aligned}
X_t^l X_t^m &= \alpha^l \alpha^m + \int_0^t \left(X_r^l b_r^m + b_r^l X_r^m \right) dr \qquad (2) \\
&\quad + \sum_{k=1}^d \int_0^t \left(X_r^l \sigma_r^{mk} + \sigma_r^{lk} X_r^m \right) dW_r^k + \sum_{k=1}^d \int_0^t a_r^{lm} \, dr.
\end{aligned}
$$

Proof of Proposition 1 (3)

Expected value for the product:

Taking expectation in (2) we get:

$$\partial_t \mathbf{E}\left[X_t^l X_t^m\right] = \mathbf{E}\left[X_t^l b_t^m + b_t^l X_t^m\right] + \mathbf{E}\left[a_t^{lm}\right].$$

Therefore:

$$\partial_t \mathbf{E}\left[X_t^l X_t^m\right]_{|t=0} = \alpha^l b^m(\alpha) + b^l(\alpha)\alpha^m + a^{lm}(\alpha).$$

Proof of Proposition 1 (4)

Product of expected values: We have

$$\mathbf{E}\left[X_t^j\right] = \alpha^j + \int_0^t \mathbf{E}\left[b_s^j\right] \, ds.$$

Therefore

$$\mathbf{E}\left[X_t^l\right] \mathbf{E}\left[X_t^m\right] = \alpha^l \alpha^m + \int_0^t \mathbf{E}\left[b_s^l\right] \, \mathbf{E}\left[X_s^m\right] \, ds + \int_0^t \mathbf{E}\left[b_s^m\right] \, \mathbf{E}\left[X_s^l\right] \, ds,$$

and differentiating:

$$\partial_t \left(\mathbf{E}\left[X_t^l\right] \mathbf{E}\left[X_t^m\right] \right)_{|t=0} = \alpha^l b^m(\alpha) + b^l(\alpha) \alpha^m.$$

Proof of Proposition 1 (5)

Infinitesimal covariance: With two previous identities,

$$\begin{aligned}
\partial_t \mathbf{Cov}\left(X_t^l, X_t^m\right)_{|t=0} &= \partial_t \left(\mathbf{E}\left[X_t^l X_t^m\right] - \mathbf{E}\left[X_t^l\right]\mathbf{E}\left[X_t^m\right]\right)_{|t=0} \\
&= \partial_t \mathbf{E}\left[X_t^l X_t^m\right]_{|t=0} - \partial_t \left(\mathbf{E}\left[X_t^l\right]\mathbf{E}\left[X_t^m\right]\right)_{|t=0} \\
&= a^{lm}(\alpha).
\end{aligned}$$

Therefore:

$$\partial_t \mathbf{Cov}\left(X_t\right)_{|t=0} = a(\alpha).$$

Geometrical Brownian motion

Proposition 4.

Let:

- W 1-dimensional Brownian motion .
- $\mu \in \mathbb{R}$ and $\sigma > 0$

We set

$$X_t = \alpha \exp\left(\mu\, t + \sigma\, W_t\right).$$

Then X is solution of (1) with $n = d = 1$ and:

$$b(x) = \left(\mu + \frac{\sigma^2}{2}\right) x, \quad \text{and} \quad \sigma(x) = \sigma\, x.$$

Geometrical Brownian motion (2)

Remarks:

- In order to show that X is solution of the equation
  $\hookrightarrow$ Apply Itô's formula.
- Exponential Brownian motion is very useful in finance (asset price):
  1. $X_t \geq 0$.
  2. Linear trend (infinitesimal drift): $\left(\mu + \frac{\sigma^2}{2}\right) X_t$.
  3. Fluctuations (infinitesimal standard deviation) proportional to X_t.

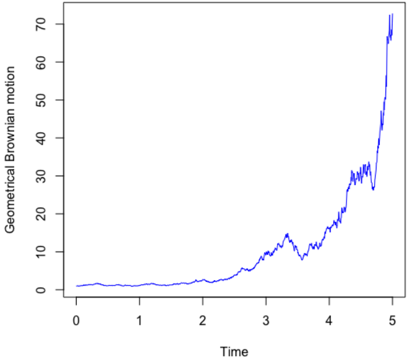
  Summarized in Black and Scholes model.

Geometrical Brownian motion: illustration

Equation: Take $\mu = 1$, $\sigma = \frac{1}{2}$, $X_0 = 1$ and

$$dX_t = \left(\mu + \frac{\sigma^2}{2}\right) X_t \, dt + \sigma X_t \, dW_t$$

Simulation:



Ornstein-Uhlenbeck process

Situation:
Velocity of a Brownian particle with friction α.

Equation:

$$dX_t = -\alpha X_t \, dt + dW_t, \qquad X_0 = a \in \mathbb{R}$$

Explicit solution:

$$X_t = e^{-\alpha t} \left(a + \sigma \int_0^t e^{\alpha s} \, dW_s \right)$$

Distribution: For $t > 0$ we have

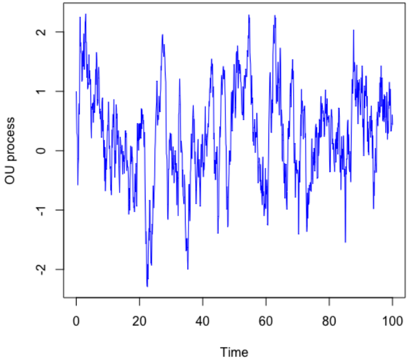
$$X_t \sim \mathcal{N}\left(a e^{-\alpha t}, \sigma_t^2 \right), \quad \text{with} \quad \sigma_t^2 = \frac{\sigma^2 (1 - e^{-2\alpha t})}{2\alpha}$$

Ornstein-Uhlenbeck: illustration

Equation: Take $\alpha = 1$ and

$$dX_t = -\alpha X_t \, dt + dW_t, \qquad X_0 = 1$$

Simulation:



Galton-Watson process

Model: We start from $n\alpha$ persons for generation $m = 0$, then

- At each generation m, individuals have i.i.d offspring
- Common law for offspring: random variable Q.
- For $n \geq 1$, sequence of number of persons at generation m:

$$\{Z_m^n;\ m \geq 0\}.$$

Assumptions on Q: We suppose

1. $\mathbf{E}[Q] = 1 + \frac{\beta}{n}$ with $\beta > 0$.
2. $\mathbf{Var}(Q) = \sigma^2$ with $\sigma^2 > 0$.
3. For all $\delta > 0$, we have $\lim_{n\to\infty} \mathbf{E}[Q^2 \mathbf{1}_{(Q>\delta n)}] = 0$

Feller diffusion

Expectation and variance computations: Thanks to offspring $\perp\!\!\!\perp$, we have

$$\mathbf{E}\left[Z_1^n \middle| Z_0^n = n\alpha\right] = n\alpha\left(1+\frac{\beta}{n}\right), \quad \mathbf{Var}\left(Z_1^n \middle| Z_0^n = n\alpha\right) = n\alpha\sigma^2.$$

Scaling: We set $X_t^n = \frac{1}{n} Z_{[nt]}^n$. Then:

$$\mathbf{E}\left[X_{1/n}^n - \alpha \middle| X_0^n = \alpha\right] = \alpha\beta\,\frac{1}{n}, \quad \mathbf{Var}\left(X_{1/n}^n \middle| X_0^n = \alpha\right) = \alpha\sigma^2\,\frac{1}{n}.$$

Limiting equation: Computing limiting drift and variance, we get

$$dX_t = \beta X_t\,dt + \sigma\sqrt{X_t}\,dW_t, \qquad X_0 = \alpha.$$

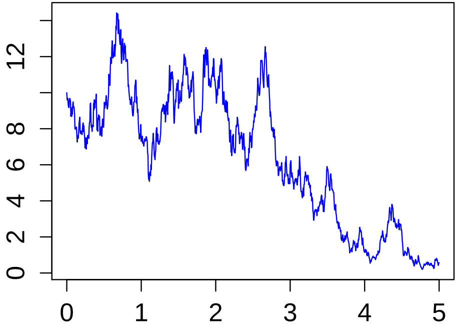
One can show that $\lim_{n\to\infty} X^n = X$ in law.

Feller diffusion: illustration

Equation:

$$dX_t = .02X_t\, dt + 2\sqrt{X_t}\, dW_t$$

Simulation:



Outline

Definition of solution

Definition 5.

We say that $(X, W, (\mathcal{F}_t)_{t\geq 0})$ is solution of (1) if:

1. W is a $\mathcal{F}_t$-Brownian motion.
2. X satisfies $X_t = \alpha + \int_0^t b(X_s)\, ds + \sum_{j=1}^d \int_0^t \sigma^j(X_s)\, dW_s^j$ for $t \geq 0$.

Definition 6.

We say that (1) admits a strong solution if:
$\hookrightarrow$ One can take $\mathcal{F}_t = \mathcal{F}_t^W$ in Definition 5.

Pathwise uniqueness

Definition 7.

Pathwise uniqueness: If X^1, X^2 are two solutions of (1) with:

1. $X_0^1 = X_0^2 = \alpha$.
2. Same Brownian motion W.

Then:

$$\mathbf{P}\left(X_t^1 = X_t^2 \text{ for all } t \geq 0\right) = 1.$$

Remark: Absence of strong solution and non pathwise uniqueness
$\hookrightarrow$ For very irregular coefficients b, σ.

Existence and uniqueness

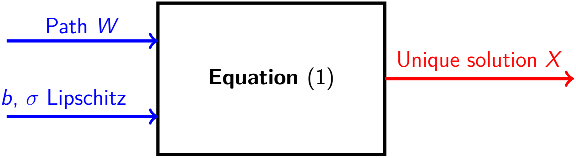
Theorem 8.

We assume that b and σ are Lipschitz fonctions:
There exist $c_\sigma, c_b > 0$ such that for all $x, y \in \mathbb{R}^n$ we have

$$|\sigma^j(x) - \sigma^j(y)| \leq c_\sigma |x - y|, \quad |b(x) - b(y)| \leq c_b |x - y|.$$

Then on every interval $[0, \tau]$, equation (1) admits:

1. A strong solution in $L^2(\Omega; \mathcal{C}([0, \tau]))$.
2. Pathwise uniqueness in $L^2(\Omega; \mathcal{C}([0, \tau]))$.



Proof: strategy

Key application: We define $\Gamma : L^2_a([0,\tau]) \to L^2_a([0,\tau])$ as:

$$\Gamma(Y) \equiv \tilde{Y}, \quad \tilde{Y}_t = \alpha + \int_0^t b(Y_s)\,ds + \sum_{j=1}^{d} \int_0^t \sigma^j(Y_s)\,dW^j_s.$$

Picard iterations: We set $X^0 \equiv \alpha$ and $X^{n+1} = \Gamma(X^n)$.

Aim: Show that:

1. X^n converges to X, where X is a strong solution.
2. Pathwise uniqueness.

Simplification in proofs:
We suppose $n = d = 1$ and $b, \sigma : \mathbb{R} \to \mathbb{R}$.

Bounds on application Γ

Lemma 9.

Let:

- $Y, Z \in L_a^2([0,\tau])$.
- $\tilde{Y} = \Gamma(Y)$, $\tilde{Z} = \Gamma(Z)$.

We assume that $c_\sigma, c_b \leq K$. Then:

$$\mathbf{E}\left[\sup_{t\leq\tau}\left|\tilde{Y}_t - \tilde{Z}_t\right|^2\right] \leq c_{K,\tau,d}\, \mathbf{E}\left[\int_0^\tau |Y_t - Z_t|^2\, dt\right].$$

with:

$$c_{K,\tau,d} = \left(2\tau d + 8d^2\right) K^2.$$

Proof

Expression for the difference: We have

$$\begin{aligned}\tilde{Y}_t - \tilde{Z}_t &= \int_0^t [b(Y_s) - b(Z_s)]\, ds + \int_0^t [\sigma(Y_s) - \sigma(Z_s)]\, dW_s \\ &\equiv J_{t,1} + J_{t,2}.\end{aligned}$$

Therefore

$$\left|\tilde{Y}_t - \tilde{Z}_t\right|^2 \leq 2(|J_{t,1}|^2 + |J_{t,2}|^2)$$

and:

$$\sup_{t\leq\tau}\left|\tilde{Y}_t - \tilde{Z}_t\right|^2 \leq 2\left(\sup_{t\leq\tau}|J_{t,1}|^2 + \sup_{t\leq\tau}|J_{t,2}|^2\right).$$

Bounds for the Lebesgue integral

Application of Jensen: We have

$$\begin{aligned} |J_{t,1}|^2 &= \left| \int_0^t [b(Y_s) - b(Z_s)] \, ds \right|^2 \\ &\leq t \int_0^t |b(Y_s) - b(Z_s)|^2 \, ds \\ &\leq \tau \int_0^\tau |b(Y_s) - b(Z_s)|^2 \, ds. \end{aligned}$$

Lipschitz property for b: We get

$$\begin{aligned} \sup_{t \leq \tau} |J_{t,1}|^2 &\leq \tau K^2 \int_0^\tau |Y_s - Z_s|^2 \, ds \\ \mathbf{E}\left[\sup_{t \leq \tau} |J_{t,1}|^2 \right] &\leq \tau K^2 \int_0^\tau \mathbf{E}\left[|Y_s - Z_s|^2 \right] ds \end{aligned}$$

Doob's maximal inequality

Proposition 10.

Let $\tau > 0$ and:

- W standard Brownian motion.
- $u \in L^2_a([0,\tau])$.
- $M_t \equiv \int_0^t u_r \, dW_r$.

Then we have:

$$\mathbf{E}\left[\sup_{t\in[0,\tau]} |M_t|^2\right] \leq 4\,\mathbf{E}\left[|M_\tau|^2\right] = 4\,\mathbf{E}\left[\int_0^\tau u_r^2 \, dr\right].$$

Bounds for the stochastic integral

Recall: $J_{t,2} = \int_0^t [\sigma(Y_s) - \sigma(Z_s)] \, dW_s$

Application of Proposition 10: We have

$$\begin{aligned} \mathbf{E}\left[\sup_{t\le\tau} |J_{t,2}|^2\right] &= \mathbf{E}\left[\sup_{t\le\tau} \left|\int_0^t [\sigma(Y_s) - \sigma(Z_s)] \, dW_s\right|^2\right] \\ &\le 4\,\mathbf{E}\left[\int_0^\tau |\sigma(Y_s) - \sigma(Z_s)|^2 \, dr\right] \end{aligned}$$

Conclusion: Lemma 9 is shown

$\hookrightarrow$ Combining inequalities for $J_{t,1}$ and $J_{t,2}$.

Bound for iterations of Γ

Lemma 11.

Let X^n Picard iterations on $[0, \tau]$:

$$X^0 \equiv \alpha \quad \text{and} \quad X^{n+1} = \Gamma(X^n).$$

We set:

$$\Delta_n(t) = \mathbf{E}\left[\sup_{s \le t} |X_s^n - X_s^{n-1}|^2\right].$$

Then there exist two constants c_1, c_2 such that:

$$\sup_{t \le \tau} \Delta_n(t) \le \frac{c_1\, c_2^n}{n!}.$$

Proof

Bound for Δ_1: We have $X_s^1 - X_s^0 = b(\alpha)s + \sigma(\alpha)W_s$. Therefore:

$$\Delta_1(t) \leq c_1\, t.$$

Induction: We have

$$X^n - X^{n-1} = \Gamma\left(X^{n-1}\right) - \Gamma\left(X^{n-2}\right).$$

According to Lemma 9, we get:

$$\Delta_n(t) \leq c_2 \int_0^t \Delta_{n-1}(s)\, ds.$$

With value of $\Delta_1(t)$, we get the result.

Convergence of X^n

Lemma 12.

Let X^n Picard iterations on $[0,\tau]$.
Then a.s X^n converges to a process X in $\mathcal{C}([0,\tau])$.

Proof

Reduction to a series convergence: We have

$$X^n = X^0 + \sum_{j=1}^{n-1} \left(X^{j+1} - X^j \right).$$

Therefore $\lim_{n\to\infty} X^n$ exists as long as $\sum_j (X^{j+1} - X^j)$ is convergent.

Series convergence:
Let $A_n = (\sup_{t\le\tau} |X_t^n - X_t^{n-1}| \ge \frac{1}{2^n})$. We will show:

$$\mathbf{P}\left(\limsup_{n\to\infty} A_n \right) = \mathbf{P}\left(A_n \text{ realized } \infty\text{-tly often} \right) = 0.$$

This entails convergence of $\sum_j (X^{j+1} - X^j)$ in $\| \cdot \|_\infty$.

Proof (2)

Application of Borel-Cantelli: We have

$$\begin{aligned} \mathbf{P}(A_n) &\leq 2^{2n}\,\mathbf{E}\left[\sup_{t\leq\tau}\left|X_t^n - X_t^{n-1}\right|^2\right] \\ &= 4^n\,\Delta_n(t) \\ &\leq \frac{4^n\,c_1\,c_2^n}{n!} = \frac{c_1\,c_3^n}{n!} \end{aligned}$$

Therefore:

$$\sum_{n\geq 1}\mathbf{P}(A_n) < \infty \quad\Longrightarrow\quad \mathbf{P}\left(\limsup_{n\to\infty} A_n\right) = 0.$$

This finishes the proof of Lemma 12.

Convergence of X^n in $L^2(\Omega)$

Lemma 13.

Let:

- X^n Picard iterations on $[0,\tau]$.
- X limit of X^n obtained in Lemma 12.
- $\|f\|_{\infty,\tau} \equiv \sup_{t\leq\tau} |f_t|$.

Then:

$$L^2(\Omega) - \lim_{n\to\infty} \|X^n - X\|_{\infty,\tau} = 0,$$

and thus:

$$L^2_a([0,\tau]) - \lim_{n\to\infty} X^n = X.$$

Proof

Notation: We set $\|Z\|_2 = \mathbf{E}^{1/2}[Z^2]$ for a real-valued r.v Z.

Cauchy sequence: We will show that

$$\lim_{m,n\to\infty} \left\| \|X^n - X^m\|_{\infty,\tau} \right\|_2 = 0. \tag{3}$$

However:

$$\begin{aligned} \left\| \|X^n - X^m\|_{\infty,\tau} \right\|_2 &\leq \left\| \sum_{k=m}^{n-1} \left\|X^{k+1} - X^k\right\|_{\infty,\tau} \right\|_2 \\ &\leq \sum_{k=m}^{n-1} \left\| \left\|X^{k+1} - X^k\right\|_{\infty,\tau} \right\|_2 \leq \sum_{k=m}^{n-1} \Delta_k^{1/2}(\tau). \end{aligned}$$

This proves (3) and Lemma 13.

Existence of a solution to (1)

Lemma 14.

Let:

- X^n Picard iterations on $[0, \tau]$.
- X limit of X^n obtained by Lemma 12.

Then X is solution of equation (1).

Proof

Strategy: We set $\tilde{X} \equiv \Gamma(X)$. We will show that $\tilde{X} = X$.

Sufficient condition: We will see that:

$$\lim_{n\to\infty} \mathbf{E}\left[\|\tilde{X} - X^{n+1}\|_{\infty,\tau}^2\right] = 0.$$

Verification: Recall that

- $X^{n+1} = \Gamma(X^n)$.
- Lemmas 9 and 13.

This yields:

$$\mathbf{E}\left[\|\tilde{X} - X^{n+1}\|_{\infty,\tau}^2\right] = \mathbf{E}\left[\|\Gamma(X) - \Gamma(X^n)\|_{\infty,\tau}^2\right] \le c\,\|X^n - X\|_{L_a^2}^2 \longrightarrow 0.$$

Gronwall's lemma

Lemma 15.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ continuous. We assume:

$$\varphi_t \leq c + d \int_0^t \varphi_s \, ds,$$

with two constants $c, d > 0$. Then we have:

$$\varphi_t \leq c \exp(dt).$$

Proof

Majorizing function: For $\varepsilon > 0$, we set $\psi_t = (c + \varepsilon)\exp(dt)$.

Comparison between φ and ψ: We assume $\tau < \infty$ with:

$$\tau = \inf\{t \geq 0;\, \varphi_t \geq \psi_t\}.$$

Then $\tau > 0$ and $\varphi_\tau = \psi_\tau$ because φ, ψ continuous.

Contradiction: We have

$$\begin{aligned}\psi_\tau &= c + \varepsilon + d\int_0^\tau \psi_s\, ds > c + d\int_0^\tau \psi_s\, ds \\ &\geq c + d\int_0^\tau \varphi_s\, ds \geq \varphi_\tau.\end{aligned}$$

Therefore $\psi_\tau > \varphi_\tau$, contradiction with $\varphi_\tau = \psi_\tau$.

Pathwise uniqueness

Lemma 16.

We consider:

- Lipschitz coefficients b, σ.
- Space of processes $L^2(\Omega; \mathcal{C}([0,\tau]))$, characterized by:

$$\|Z\|^2_{L^2(\Omega;\mathcal{C}([0,\tau]))} = \mathbf{E}\left[\sup_{t\leq\tau} |Z_t|^2\right].$$

Then we have pathwise uniqueness for equation (1)
$\hookrightarrow$ In $L^2(\Omega; \mathcal{C}([0,\tau]))$.

Proof

Aim: Let $(X^1, W, \mathcal{F}_t^1)$, $(X^2, W, \mathcal{F}_t^2)$ two solutions
$\hookrightarrow$ We wish to show that $X^1 = X^2$.

Filtrations: Let $\mathcal{F}_t = \mathcal{F}_t^1 \vee \mathcal{F}_t^2$
$\hookrightarrow$ We have X^1, X^2 adapted for $\mathcal{F}_t$
$\hookrightarrow$ Estimates for stochastic integrals can be applied to $X^1 - X^2$.

Application of Lemma 9:
We set $\varphi_t = \|X^1 - X^2\|^2_{L^2(\Omega;\mathcal{C}([0,\tau]))}$. Then:

$$\varphi_t = \|\tilde{X}^1 - \tilde{X}^2\|^2_{L^2(\Omega;\mathcal{C}([0,t]))} \leq d \int_0^t \varphi_s \, ds,$$

with $d = c_{K,\tau,d}$. Therefore $\varphi \equiv 0$ and $X^1 = X^2$.

More existence and uniqueness results

Extensions:

We have existence and uniqueness for (1) in following situations:

1. Coefficients $b(s,x)$ and $\sigma(s,x)$ with uniform Lipschitz conditions:
$$|b(s,x)-b(s,y)|+|\sigma(s,x)-\sigma(s,y)|\leq c\,|x-y|.$$
2. Coefficients b,σ locally Lipschitz with linear growth:
$$\begin{aligned}|b(x)-b(y)|+|\sigma(x)-\sigma(y)| &\leq c_n\,|x-y|, \text{ for } |x|,|y|\leq n.\\ |b(x)|+|\sigma(x)| &\leq c\,(1+|x|).\end{aligned}$$
3. Case $d=1$ and
   - b Lipschitz.
   - σ Hölder-continuous Hölder exponent $\alpha\geq 1/2$.

Outline

Definition of fBm

Complete probability space: $(\Omega, \mathcal{F}, \mathbf{P})$

Definition 17.

A 1-d fBm is a continuous process $B = \{B_t;\ t \geq 0\}$ such that $B_0 = 0$ and for $H \in (0,1)$:

- B is a centered Gaussian process
- $\mathbf{E}[B_t B_s] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H})$

d-dimensional fBm: $B = (B^1, \ldots, B^d)$, with B^i independent 1-d fBm

Variance of the increments:

$$\mathbf{E}[|\delta B^j_{st}|^2] \equiv \mathbf{E}[|B_t - B_s|^2] = |t-s|^{2H}$$

Kolmogorov criterion

Notation: If $f : [0,\tau] \to \mathbb{R}^d$ is a function, we shall denote:

$$\delta f_{st} = f_t - f_s, \quad \text{and} \quad \|f\|_\mu = \sup_{s,t\in[0,\tau]} \frac{|\delta f_{st}|}{|t-s|^\mu}$$

Theorem 18.

Let $X = \{X_t;\, t \in [0,\tau]\}$ be a process defined on $(\Omega, \mathcal{F}, \mathbf{P})$, such that

$$\mathbf{E}\left[|\delta X_{st}|^\alpha\right] \le c|t-s|^{1+\beta}, \quad \text{for} \quad s,t\in[0,\tau],\ c,\alpha,\beta>0$$

Then there exists a modification $\hat{X}$ of X such that almost surely $\hat{X} \in \mathcal{C}_1^\gamma$ for any $\gamma < \beta/\alpha$, i.e $\mathbf{P}(\omega;\, \|\hat{X}(\omega)\|_\gamma < \infty) = 1$.

FBm regularity

Proposition 19.

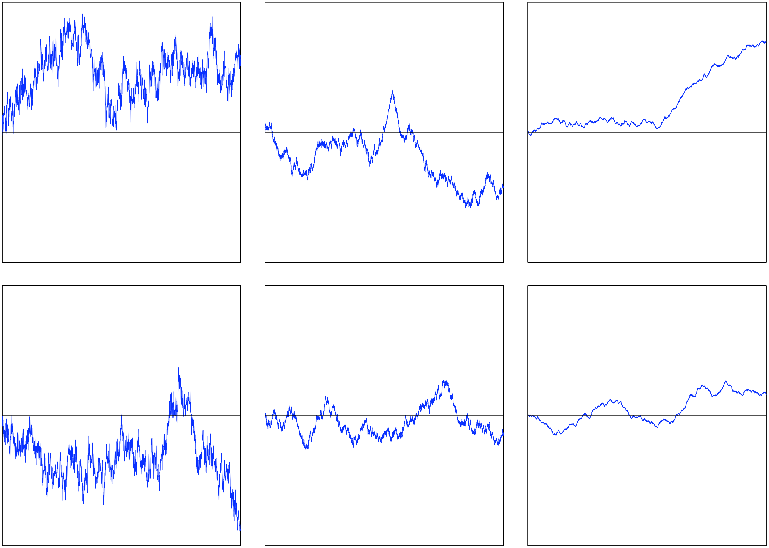
FBm $B \equiv B^H$ is γ-Hölder continuous for all $\gamma < H$, up to modification.

Proof: We have $\delta B_{st} \sim \mathcal{N}(0, |t-s|^{2H})$. Thus for $n \geq 1$,

$$\mathbf{E}\left[|\delta B_{st}|^{2n}\right] = c_n |t-s|^{2Hn} \quad \text{i.e} \quad \mathbf{E}\left[|\delta B_{st}|^{2n}\right] = c_n |t-s|^{1+(2Hn-1)}$$

Kolmogorov: B is γ-Hölder for $\gamma < (2Hn-1)/2n = H - 1/(2n)$.
Proof finished by letting $n \to \infty$.

Examples of fBm paths



$H = 0.3$ $H = 0.5$ $H = 0.7$

Some properties of fBm

Proposition 20.

Let B be a fBm with parameter H. Then:

1. $\{a^{-H} B_{at};\ t \geq 0\}$ is a fBm (scaling)
2. $\{B_{t+h} - B_h;\ t \geq 0\}$ is a fBm (stationarity of increments)
3. B is not a semi-martingale unless $H = 1/2$
   And B is nowhere differentiable a.s

Proof of claim 3:

If B were a semi-martingale, we would get on $[0,1]$:

$$\mathbf{P} - \lim_{n\to\infty} \sum_{i=1}^{n} (B_{i/n} - B_{(i-1)/n})^2 = \langle B \rangle_1 ,$$

were $\langle B \rangle$ is the (non trivial) quadratic variation of B.

We will show that $\langle B \rangle$ is trivial (0 or ∞) whenever $H \neq 1/2$.

Proof of claim 3 (2)

Define

$$V_{n,2} = \sum_{i=1}^{n} |B_{i/n} - B_{(i-1)/n}|^2, \quad \text{and} \quad Y_{n,2} = n^{2H-1} V_{n,2}.$$

By scaling properties, we have:

$$Y_{n,2} \stackrel{(d)}{=} \hat{Y}_{n,2}, \quad \text{with} \quad \hat{Y}_{n,p} = n^{-1} \sum_{i=1}^{n} |B_i - B_{i-1}|^2.$$

The sequence $\{B_i - B_{i-1};\, i \geq 1\}$ is stationary and mixing
$\Rightarrow \hat{Y}_{n,2}$ converges $\mathbf{P} - a.s$ and in L^1 towards $\mathbf{E}[|B_1 - B_0|^2]$
$\Rightarrow \mathbf{P} - lim_{n\to\infty} Y_{n,2} = E[|B_1|^2]$
$\Rightarrow \mathbf{P} - lim_{n\to\infty} V_{n,2} = 0$ if $2H > 1$, ∞ if $2H < 1$

Outline

Strategy for $H > 1/2$

- Generally speaking, take advantage of two aspects of fBm:
  - Gaussianity
  - Regularity

  For $H > 1/2$, regularity is almost sufficient
- Notation: $\mathcal{C}_1^\gamma = \mathcal{C}_1^\gamma(\mathbb{R}) \equiv \gamma$-Hölder functions of 1 variable
- If $H > 1/2$, $B \in \mathcal{C}_1^\gamma$ for any $1/2 < \gamma < H$ a.s
- We shall try to solve our equation in a pathwise manner

Pathwise strategy

Aim: Let x be a function in $\mathcal{C}_1^{\gamma}$ with $\gamma > 1/2$. We wish to define and solve an equation of the form:

$$y_t = a + \int_0^t \sigma(y_s)\, dx_s \tag{4}$$

Steps:

- Define an integral $\int z_s\, dx_s$ for $z \in \mathcal{C}_1^{\kappa}$, with $\kappa + \gamma > 1$
- Solve (4) through fixed point argument in $\mathcal{C}_1^{\kappa}$ with $1/2 < \kappa < \gamma$

Remark: We treat a real case and $b \equiv 0$ for notational sake.
$\hookrightarrow$ Extensions to dimension d by adding indices.

Particular Riemann sums

Aim: Define $\int_0^1 z_s\, dx_s$ for $z \in \mathcal{C}_1^\kappa, x \in \mathcal{C}_1^\gamma$, with $\kappa + \gamma > 1$

Dyadic partition: set $t_i^n = i/2^n$, for $n \geq 0$, $0 \leq i \leq 2^n$

Associated Riemann sum:

$$I_n \equiv \sum_{i=0}^{2^n-1} z_{t_i^n}[x_{t_{i+1}^n} - x_{t_i^n}] = \sum_{i=0}^{2^n-1} z_{t_i^n}\, \delta x_{t_i^n t_{i+1}^n}.$$

Question: Can we define $\mathcal{J}_{01}(z\, dx) \equiv \lim_{n\to\infty} I_n$?

Possibility: Control $|I_{n+1} - I_n|$ and write (if the series is convergent):

$$\mathcal{J}_{01}(z\, dx) = I_0 + \sum_{n=0}^{\infty}(I_{n+1} - I_n).$$

Control of $I_{n+1} - I_n$

We have:

$$\begin{aligned} I_n &= \sum_{i=0}^{2^n-1} z_{t_i^n}\, \delta x_{t_i^n t_{i+1}^n} = \sum_{i=0}^{2^n-1} z_{t_{2i}^{n+1}} \left[\delta x_{t_{2i}^{n+1} t_{2i+1}^{n+1}} + \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}} \right] \\ I_{n+1} &= \sum_{i=0}^{2^n-1} \left[z_{t_{2i}^{n+1}}\, \delta x_{t_{2i}^{n+1} t_{2i+1}^{n+1}} + z_{t_{2i+1}^{n+1}}\, \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}} \right] \end{aligned}$$

Therefore:

$$\begin{aligned} |I_{n+1} - I_n| &= \left| \sum_{i=0}^{2^n-1} \delta z_{t_{2i}^{n+1} t_{2i+1}^{n+1}}\, \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}} \right| \\ &\le \sum_{i=0}^{2^n-1} \|z\|_\kappa |t_{2i+1}^{n+1} - t_{2i}^{n+1}|^\kappa\ \|x\|_\gamma |t_{2i+2}^{n+1} - t_{2i+1}^{n+1}|^\gamma \\ &= \frac{\|z\|_\kappa \|x\|_\gamma}{2^{\kappa+\gamma} 2^{n(\kappa+\gamma-1)}} \end{aligned}$$

Definition of the integral

We have seen: for $\alpha \equiv \kappa + \gamma - 1 > 0$ and $n \geq 0$:

$$|I_{n+1} - I_n| \leq \frac{c_{x,z}}{2^{\alpha n}}$$

Series convergence:
Obviously, $\sum_{n=0}^{\infty}(I_{n+1} - I_n)$ is a convergent series
$\hookrightarrow$ yields definition of $\mathcal{J}_{01}(z\,dx)$, and more generally: $\mathcal{J}_{st}(z\,dx)$

Remark:
One should consider more general partitions π, with $|\pi| \to 0$
$\hookrightarrow$ C.f Lejay (Séminaire 37)

Young integral

Proposition 21.

Let

- $z \in \mathcal{C}_1^{\kappa}([0,\tau])$, $x \in \mathcal{C}_1^{\gamma}([0,\tau])$
- $\kappa + \gamma > 1$, and $0 \leq s < t \leq T$
- $(\pi_n)_{n \geq 0}$ a sequence of partitions of $[s,t]$ such that $\lim_{n\to\infty} |\pi_n| = 0$
- I_n corresponding Riemann sums

Then:

1. I_n converges to an element $\mathcal{J}_{st}(z\,dx)$
2. The limit does not depend on the sequence $(\pi^n)_{n\geq 0}$
3. Integral linear in z, and coincides with Riemann's integral for smooth z, x

A bound for Young integrals

Theorem 22.

Let $f \in \mathcal{C}_1^\kappa, g \in \mathcal{C}_1^\gamma$, with $\kappa + \gamma > 1$. Then:

1. If $0 \leq s < u < t \leq T$, we have
$\mathcal{J}_{st}(z\, dx) = \mathcal{J}_{su}(z\, dx) + \mathcal{J}_{ut}(z\, dx)$
2. Generalized integral $\mathcal{J}(f\, dg)$ satisfies:

$$|\mathcal{J}_{st}(f\, dg)| \leq \|f\|_\infty \|g\|_\gamma |t-s|^\gamma + c_{\gamma,\kappa} \|f\|_\kappa \|g\|_\gamma |t-s|^{\gamma+\kappa}.$$

Outline

Pathwise strategy (repeated)

Aim: Let x be a function in $\mathcal{C}_1^{\gamma}$ with $\gamma > 1/2$. We wish to define and solve an equation of the form:

$$y_t = a + \int_0^t \sigma(y_s)\, dx_s \tag{5}$$

Steps:

- Define an integral $\int z_s\, dx_s$ for $z \in \mathcal{C}_1^{\kappa}$, with $\kappa + \gamma > 1$
- Solve (5) through fixed point argument in $\mathcal{C}_1^{\kappa}$ with $1/2 < \kappa < \gamma$

Remark: We treat a real case and $b \equiv 0$ for notational sake.

Existence-uniqueness result

Theorem 23.

Consider

- Noise: $x \in \mathcal{C}_1^\gamma \equiv \mathcal{C}_1^\gamma([0,\tau])$, with $\gamma > 1/2$
- Coefficient: $\sigma : \mathbb{R} \to \mathbb{R}$ a C_b^2 function
- Equation: $\delta y = \mathcal{J}(\sigma(y)\, dx)$, and $y_0 = a$.

Then:

1. Our equation admits a unique solution y in $\mathcal{C}_1^\kappa$ for any $1/2 < \kappa < \gamma$.
2. Application $(a, x) \mapsto y$ is continuous from $\mathbb{R} \times \mathcal{C}_1^\gamma$ to $\mathcal{C}_1^\kappa$.

Fixed point: strategy

A map on a small interval:
Consider an interval $[0, \tau]$, with τ to be determined later

Consider κ such that $1/2 < \kappa < \gamma < 1$

In this interval, consider $\Gamma : \mathcal{C}_1^{\kappa}([0, \tau]) \to \mathcal{C}_1^{\kappa}([0, \tau])$ defined by:
$\Gamma(z) = \hat{z}$, with $\hat{z}_0 = a$, and for $s, t \in [0, \tau]$:

$$\delta \hat{z}_{st} = \int_s^t \sigma(z_r) dx_r = \mathcal{J}_{st}(\sigma(z)\, dx)$$

Aim: See that for a small enough τ, the map Γ is a contraction
$\hookrightarrow$ our equation admits a unique solution in $\mathcal{C}_1^{\kappa}([0, \tau])$