Outline

A motivating example: generative AI

Deep generative models: Often in the headlines for deep fakes
$\hookrightarrow$ Also useful in physics and computational chemistry

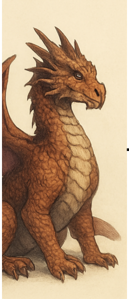
Our guiding example: Continuous normalizing flows

Three steps:

1. Sample $\mathbf{x}_0$ according to the distribution μ_0 to be faked
2. Add noise to $\mathbf{x}_0$, thanks to a simple enough equation
   $\hookrightarrow$ This produces a process $\mathbf{X}_t$
3. Reverse time in the equation, starting from $\mathbf{X}_1$
   $\hookrightarrow$ This produces a new sample $\hat{\mathbf{X}}_1 \sim \mu_0$

$\hat{X}_t = X_{1-t}$

Picture of the algorithm



Forward SDE

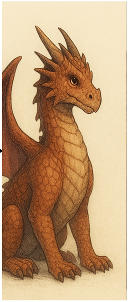
data $\rightarrow$ noisy data

$$\mathrm{d}\mathbf{X} = f(t, \mathbf{X})\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W$$

Backward SDE

noisy data $\rightarrow$ new sample

$$\mathrm{d}\hat{\mathbf{X}} = f(t, \hat{\mathbf{X}})\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W - \sigma^2(t) \times (\text{score})\,\mathrm{d}t$$



Successfully implemented in numerous domains

Forward- eq $\quad i = 1, \dots, d$

$$X_t^i = x_0^i + \int_0^t f(s, X_s^i)\, ds + \int_0^t \sigma(s)\, \underbrace{dW^i(s)}$$

Brownian motion

Nowhere differentiable

stochastic integral

stochastic diff. equation

Backward dynamics

$$\hat{X}_t = X_{1-t}$$

density of the r.v. X_s

$$\hat{X}_t = X_1 + \int_0^t f(s, \hat{X}_s)\, ds - \int_0^t \sigma^2(s)\, \nabla_x \ln\big(\rho_s(\hat{X}_s)\big)\, ds + \int_0^t \sigma(s)\, dW_s$$

Chapman - Kolmogorov eq
(PDE)

Adding noise: the forward equation

Noise addition: For $i = 1, \ldots, d$, one solves

$$\mathbf{X}_t^i = \mathbf{x}_0^i + \int_0^t f\left(s, \mathbf{X}_s^i\right) \mathrm{d}s + \int_0^t \sigma(s) \, \mathrm{d}\mathbf{W}_s^i$$

Ingredients:

- $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0,1] \to \mathbb{R}$
  $\hookrightarrow$ parameters of the algorithm.
- $\mathbf{W}$ is a d-dimensional Wiener process.

Remark: The dimension only enters through the initial data $\mathbf{x}_0$
$\hookrightarrow$ We may focus on a generic coordinate $X_t = \mathbf{X}_t^i$

Reversing time: the backward equation

Reversed process: $\hat{X}_t = X_{1-t}$ solves

$$\hat{X}_t = X_1 + \int_0^t \left[f\left(s, \hat{X}_s\right) - \sigma^2(s)\, \nabla_x \ln\left(p_s(\hat{X}_s)\right)\right] \mathrm{d}s + \int_0^t \sigma(s)\, \mathrm{d}\hat{W}_s$$

Notation:

- p_s is the density of the random variable X_s.
- $\hat{W}$ is another Brownian motion.

Conclusion: One gets $\hat{X}_1 \stackrel{(d)}{=} x_0$
$\hookrightarrow$ The deep fake is produced

Aim of the course

Observation: The two equations above involve

- A Brownian motion W.
- An integral $\int \sigma(s)\,\mathrm{d}W_s$ with respect to W.
- A stochastic differential equation, forward and backward in time.
- A score function $\nabla_x \ln p_s$, estimated by a neural network.

Main goal of the course:
$\hookrightarrow$ Understand those objects and their properties

Similar objects show up in
$\hookrightarrow$ control, reinforcement learning, signatures

Brief outline of the course

Chapters covered:

1. Basics on Brownian motion
2. Ito's formula
3. Stochastic differential equations
4. Reversed diffusion processes and generative AI algorithms
5. Stochastic control and neural networks
6. Rough paths and the signature method for machine learning
7. Relaxed control and reinforcement learning

↳ Project (?)