

# Elementary processes

## Definition 1

Let:

- $(\Omega, \mathcal{F}, \mathbf{P})$  probability space.
- $u = \{u_t; t \in [0, \tau]\}$  process defined on  $(\Omega, \mathcal{F}, \mathbf{P})$

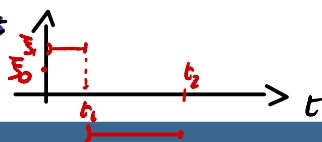
Process  $u$  is called simple if it can be decomposed as:

$$u_t = \xi_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{N-1} \xi_i \mathbf{1}_{(t_i, t_{i+1}]}(t),$$

with

- $N \geq 1$ ,  $(t_1, \dots, t_N)$  partition of  $[0, \tau]$  with  $t_1 = 0, t_N = \tau$
- $\xi_i \in \mathcal{F}_{t_i}$  and  $|\xi_i| \leq c$  with  $c > 0$  ( $c$  non random)

# Integral of a simple process



## Definition 2

Let  $u$  simple process. Let  $s, t \in [0, \tau]$  such that  $s \leq t$  and:

$$t_m < s \leq t_{m+1}, \quad \text{and} \quad t_n < t \leq t_{n+1}, \quad m \leq n$$

We define  $\mathcal{I}_{st}(u dW) = " \int_s^t u_w dW_w "$  as:

$$\mathcal{I}_{st}(u dW) = \xi_m \delta W_{st_{m+1}} + \sum_{i=m+1}^{n-1} \xi_i \delta W_{t_i t_{i+1}} + \xi_n \delta W_{t_n t}$$

**Remark:** For simple processes

$\hookrightarrow$  Stochastic integral and Riemann integral coincide.

# Integral for simple processes: properties

## Proposition 3

Consider:

- $u$  a simple process,  $v$  another simple process
- $\mathcal{I}_{st}(u dW)$  its stochastic integral.

Then for  $\alpha, \beta \in \mathbb{R}$ :

- ①  $\mathcal{I}_{tt}(u dW) = 0 = \int_t^t u_r dW_r$
- ②  $\mathcal{I}_{st}((\alpha u + \beta v) dW) = \alpha \mathcal{I}_{st}(u dW) + \beta \mathcal{I}_{st}(v dW)$
- ③  $\mathbf{E}[\mathcal{I}_{st}(u dW) | \mathcal{F}_s] = 0$ , i.e martingale property for  $t \mapsto \int_s^t u_r dW_r$
- ④  $\mathbf{E}[(\mathcal{I}_{st}(u dW))^2] = \int_s^t \mathbf{E}[u_\tau^2] d\tau$
- ⑤  $\mathbf{E}[(\mathcal{I}_{st}(u dW))^2 | \mathcal{F}_s] = \int_s^t \mathbf{E}[u_\tau^2 | \mathcal{F}_s] d\tau$

Proof of (u simple process)

$$\begin{aligned} & E \left[ \left( \int_s^t (u \, dw) \right)^2 \mid \mathcal{F}_s \right] \\ &= \int_s^t E \left[ u_r^2 \mid \mathcal{F}_s \right] \, dr \end{aligned}$$

Simplification: take  $s = t_m$ ,  $t = t_n$   
with  $m < n$ . Then

$$\int_{t_m}^{t_n} (u \, dw) = \int_{t_m}^{t_n} u_r \, dw_r$$

$$= \sum_{k=m}^{n-1} \bar{u}_k \Delta W_{t_k, t_{k+1}}$$

(here  $W_t \in \mathbb{R}$   
 $\bar{u}_k \in \mathbb{R}$ )

Thus

$$\begin{aligned} & \mathbb{E} \left[ \left( \int_{t_m}^{t_n} (u \, dW) \right)^2 \mid \mathcal{F}_{t_m} \right] \\ &= \mathbb{E} \left[ \left( \sum_{k=m}^{n-1} \xi_k \, \delta W_{t_k, t_{k+1}} \right)^2 \mid \mathcal{F}_{t_m} \right] \\ &= \sum_{k=m}^{n-1} \mathbb{E} \left[ \xi_k^2 (\delta W_{t_k, t_{k+1}})^2 \mid \mathcal{F}_{t_m} \right] \\ &+ 2 \sum_{m \leq k < l \leq n-1} \mathbb{E} \left[ \xi_k \xi_l \, \delta W_{t_k, t_{k+1}} \delta W_{t_l, t_{l+1}} \mid \mathcal{F}_{t_m} \right] \end{aligned}$$

Crossed term

for  $m \leq k < \ell \leq n-1$

Hyp  $\xi_j \in \mathcal{F}_{t_j}$

$$\begin{aligned} & \mathbb{E} [\xi_k \xi_\ell \delta W_{t_k t_{k+1}} \delta W_{t_\ell t_{\ell+1}} | \mathcal{F}_m] \\ &= \mathbb{E} \left\{ \mathbb{E} [\underbrace{\xi_k}_{\in \mathcal{F}_{t_k}} \underbrace{\xi_\ell}_{\in \mathcal{F}_{t_\ell}} \underbrace{\delta W_{t_k t_{k+1}}}_{\in \mathcal{F}_{t_k}} \underbrace{\delta W_{t_\ell t_{\ell+1}}}_{\perp \mathcal{F}_{t_\ell}} | \mathcal{F}_{t_\ell}] | \mathcal{F}_m \right\} \\ &= \mathbb{E} \left\{ \xi_k \xi_\ell \delta W_{t_k t_{k+1}} \underbrace{\mathbb{E} [\delta W_{t_\ell t_{\ell+1}}]}_{=0} | \mathcal{F}_m \right\} = 0 \end{aligned}$$

If  $X \perp g$ ,  $Y \in g$ ,  $\mathbb{E}[XY | g] = \mathbb{E}[X] Y$

If  $g_1 \subset g_2$   $\mathbb{E} \{ \mathbb{E}[X | g_2] | g_1 \}$   
 $= \mathbb{E}[X | g_1]$

Quadratic terms

For  $k \geq m$

$$\mathbb{E} \left[ \sum_k^2 (\delta W_{t_k t_{k+1}})^2 \mid \mathcal{F}_{t_m} \right]$$

$$= \mathbb{E} \left\{ \mathbb{E} \left[ \underbrace{\sum_k^2}_{\in \mathcal{F}_{t_k}} \underbrace{(\delta W_{t_k t_{k+1}})^2}_{\perp \mathcal{F}_{t_k}} \mid \mathcal{F}_{t_k} \right] \mid \mathcal{F}_{t_m} \right\}$$

$$= \mathbb{E} \left\{ \sum_k^2 \mathbb{E} [(\delta W_{t_k t_{k+1}})^2] \mid \mathcal{F}_{t_m} \right\}$$

$$= \mathbb{E} \left\{ \sum_k^2 \mid \mathcal{F}_{t_m} \right\} (t_{k+1} - t_k)$$

## Summarizing

$$\mathbb{E} \left[ \left( \int_{t_m}^{t_n} u \, dw \right)^2 \mid \mathcal{F}_{t_m} \right]$$

$$= \sum_{k=m}^{n-1} \mathbb{E} \left\{ \tilde{\Sigma}_k^2 \mid \mathcal{F}_{t_m} \right\} (t_{k+1} - t_k)$$

(linearity of  $\mathbb{E}[\cdot | \mathcal{F}]$ )

$$= \mathbb{E} \left\{ \sum_{k=m}^{n-1} \tilde{\Sigma}_k^2 (t_{k+1} - t_k) \mid \mathcal{F}_{t_m} \right\}$$

$$= \mathbb{E} \left\{ \int_{t_m}^{t_n} u_r^2 \mid \mathcal{F}_{t_m} \right\}$$

$$= \int_{t_m}^{t_n} \mathbb{E} \{ u_r^2 \mid \mathcal{F}_{t_m} \} \, dr$$

Rmk: There were 5 prop. Try to prove ③

Remark We have seen (for  $u$  simple)

$$\mathbb{E} \left[ \mathbb{E} \left\{ \left( \int_s^t u_r dw_r \right)^2 \mid \mathcal{F}_s \right\} \right]$$

$$= \mathbb{E} \left[ \int_s^t \mathbb{E} [u_r^2 \mid \mathcal{F}_s] dr \right]$$

$$\Rightarrow \begin{aligned} & \mathbb{E} \left\{ \left( \int_s^t u_r dw_r \right)^2 \right\} \\ &= \int_s^t \mathbb{E} [u_r^2] dr \end{aligned} \quad (4)$$

$$\mathbb{E} [\mathbb{E} [X \mid \mathcal{G}]] = \mathbb{E} [X]$$

# Proof of point 5

If  $i < j$ , then (independence of increments of  $B$ )

$$\begin{aligned} & \mathbf{E} \left[ \xi_i \delta W_{t_i t_{i+1}} \xi_j \delta W_{t_j t_{j+1}} \middle| \mathcal{F}_{t_j} \right] \\ &= \xi_i \xi_j \delta W_{t_i t_{i+1}} \mathbf{E} \left[ \delta W_{t_j t_{j+1}} \middle| \mathcal{F}_{t_j} \right] \\ &= \xi_i \xi_j \delta W_{t_i t_{i+1}} \mathbf{E} \left[ \delta W_{t_j t_{j+1}} \right] \\ &= 0 \end{aligned}$$

## Proof of point 5 (2)

Thus:

$$\begin{aligned} & \mathbf{E}[(\mathcal{I}_{st}(u \, dW))^2 | \mathcal{F}_s] \\ &= \mathbf{E} \left[ \left( \xi_m \delta W_{st_m} + \sum_{i=m+1}^{n-1} \xi_i \delta W_{t_i t_{i+1}} + \xi_n \delta W_{t_n t} \right)^2 \middle| \mathcal{F}_s \right] \\ &= \mathbf{E} \left[ \xi_m^2 \delta W_{st_m}^2 + \sum_{i=m+1}^{n-1} \xi_i^2 \delta W_{t_i t_{i+1}}^2 + \xi_n^2 \delta W_{t_n t}^2 \middle| \mathcal{F}_s \right] \\ &= \mathbf{E}[\xi_m^2 | \mathcal{F}_s](t_m - s) + \sum_{i=m+1}^{n-1} \mathbf{E}[\xi_i^2 | \mathcal{F}_s](t_{i+1} - t_i) + \mathbf{E}[\xi_n^2 | \mathcal{F}_s](t - t_n) \\ &= \int_s^t \mathbf{E}[u_r^2 | \mathcal{F}_s] \, dr \end{aligned}$$

# Space $L_a^2$

## Definition 4

We denote by  $L_a^2([0, \tau])$  the set of process  $u$  such that:

- ①  $u$  square integrable
- ②  $u$  left or right continuous
- ③  $u_t \in \mathcal{F}_t$

The norm on  $L_a^2$  is defined by:

$$\|u\|_{L_a^2}^2 \equiv \int_0^\tau \mathbf{E} [u_s^2] ds$$

*appears in prop ④*

**Remark:** Condition  $u_t \in \mathcal{F}_t$  **and**  $u$  left continuous

↪ Ensures **predictable property** for  $u$

↪ See discrete time stochastic integral

# Density of simple processes in $L_a^2 \rightarrow$ adapted

## Proposition 5

Let  $u \in L_a^2$ .

There exists a sequence  $(u^n)_{n \geq 0}$  of simple processes such that:

$$\lim_{n \rightarrow \infty} \|u - u^n\|_{L_a^2} = 0.$$

# Concrete approximation

**Generic partition:** We consider

- $\pi = \{s_0, \dots, s_n\}$  with  $0 = s_0 < \dots < s_n = t$
- $|\pi| = \max\{|s_{j+1} - s_j|; 0 \leq j \leq n-1\}$

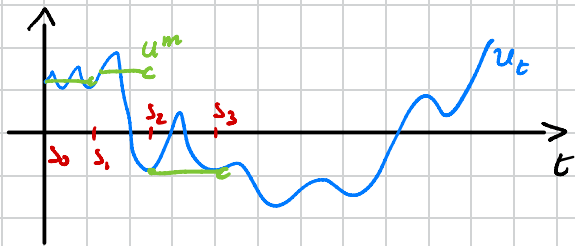
## Proposition 6

*Let:*

- $u \in L_a^2$  such that  $|u_t| \leq M < \infty$  for all  $t \leq \tau$  a.s
- Assume  $u$  is continuous (almost surely)
- $\{\pi^m; m \geq 1\}$  sequence of partitions of  $[0, \tau]$   
 $\hookrightarrow$  such that  $\lim_{m \rightarrow \infty} |\pi_m| = 0$
- $u^m \equiv \sum_{s_j \in \pi^m} u_{s_j} \mathbf{1}_{(s_j, s_{j+1}]}$

*Then we have:*

$$\lim_{m \rightarrow \infty} \|u - u^m\|_{L_a^2} = 0$$



Expression:  $u_t^m = \sum u_{s_j} 1_{[s_j, s_{j+1})}(t)$

Then

(1) We have assumed  $u$  to be continuous a.s.  
Therefore, we have

$$(u_t^m \rightarrow u_t \quad \forall t \in [0, z]) \quad \text{a.s.}$$

(2) Difference in  $L_a^2$

We have:  
 $|u_t| \leq M$   
 $|u_t^m| \leq M$

$$\|u^m - u\|_{L_a^2}^2 \rightarrow 0 \text{ a.s.}$$

def

$$= \int_0^T \mathbb{E}[(u_t - u_t^m)^2] dt$$

$\rightarrow 0$  by dominated convergence

# Proof

Expression for  $u^m$ :

$$u_t^m = \sum_{s_j \in \pi^m} u_{s_j} \mathbf{1}_{(s_j, s_{j+1}]}(t)$$

Properties of  $u^m$ :

- ① Almost surely:  $\lim_{m \rightarrow \infty} u_t^m = u_t$  for all  $t \in [0, \tau]$
- ②  $|u_t^m| \leq Z_t$  with  $Z_t \equiv M$
- ③  $Z \in L^2(\Omega \times [0, \tau])$

Convergence of  $u^m$ : by dominated convergence,

$$\lim_{m \rightarrow \infty} \int_0^\tau \mathbf{E} [|u_t - u_t^m|^2] = 0$$

# Stochastic integral: extended definition

## Proposition 7

Let:

- $u \in L_a^2$
- $(u^n)_{n \geq 0}$  sequence of simple processes such that  $\lim_{n \rightarrow \infty} \|u - u^n\|_{L_a^2} = 0$

Then for  $s < t$ :

- The sequence  $(\mathcal{J}_{st}(u^n dW))_{n \geq 0}$  converges in  $L^2(\Omega)$
- Its limit does not depend on the sequence  $(u^n)_{n \geq 0}$

"Proof" Take  $(u^n)_{n \geq 1}$  simple processes  
s.t.  $u^n \rightarrow u$  in  $L^2_{\mathcal{A}}$ .

We prove that  $\int_0^t u_r^n dw_r$  is  
a Cauchy sequence in  $L^2(\mathcal{R})$ :

$$\begin{aligned} & \mathbb{E} \left[ \left( \int_0^t u_r^n dw_r - \int_0^t u_r^m dw_r \right)^2 \right] \\ &= \mathbb{E} \left[ \left( \int_0^t (u_r^n - u_r^m) dw_r \right)^2 \right] \\ &\stackrel{(4)}{=} \mathbb{E} \left\{ \int_0^t (u_r^n - u_r^m)^2 dr \right\} \rightarrow 0 \quad (u^n \text{ Cauchy in } L^2_{\mathcal{A}}) \end{aligned}$$

Thus  $\left( \int_0^t u_r^n dw_r \right)_{n \geq 1}$  converges in  $L^2(\mathcal{R})$

# Proof

Cauchy sequence  $u^n$ : We know that:

$$\|u^n - u^m\|_{L^2_s} = \int_s^t \mathbf{E} \left[ (u_w^n - u_w^m)^2 \right] dw \xrightarrow{n \rightarrow \infty} 0.$$

Cauchy sequence  $\mathcal{J}_{st}(u^n)$ : According to property:

$$\mathbf{E} \left[ |\mathcal{J}_{st}((u^n - u^m) dW)|^2 \right] = \int_s^t \mathbf{E} \left[ |u_w^n - u_w^m|^2 \right] dw$$

we have

$$(Z_n)_{n \geq 0} \equiv (\mathcal{J}_{st}(u^n))_{n \geq 0}$$

is a Cauchy sequence in  $L^2(\Omega)$ .