

Outline

1 Introduction and examples

2 Existence and uniqueness

Aim

Coefficients: We consider

- $\alpha \in \mathbb{R}^n$ and $b, \sigma^1, \dots, \sigma^d : \mathbb{R}^n \rightarrow \mathbb{R}^n$.
- We denote: $\sigma = (\sigma^1, \dots, \sigma^d) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$
- W , d -dimensional Brownian motion.

Equation: We wish to solve

$$dX_s = b(X_s) ds + \sum_{j=1}^d \sigma^j(X_s) dW_s^j.$$

Integral form: With Itô's integral, *implicit: $\sigma^j(X_s) \in \mathcal{L}_a^2$*

$$X_t = \alpha + \int_0^t b(X_s) ds + \sum_{j=1}^d \int_0^t \overbrace{\sigma^j(X_s)}^{\text{implicit}} dW_s^j. \quad (1)$$

Infinitesimal drift and covariance $\text{If } X, Y \in \mathbb{R}$

$$\begin{aligned}\text{cov}(X, Y | \mathcal{F}) &= \mathbb{E}\{(X - \mathbb{E}[X | \mathcal{F}])(Y - \mathbb{E}[Y | \mathcal{F}]) | \mathcal{F}\} \\ &= \mathbb{E}[XY | \mathcal{F}] - \mathbb{E}[X | \mathcal{F}] \mathbb{E}[Y | \mathcal{F}]\end{aligned}$$

Proposition 1

Let b, σ bounded, X solution of (1). Then:

$$\partial_t \mathbf{E}[X_t | \mathcal{F}_s]_{|_{t=s}} = b(X_s), \quad \partial_t \mathbf{Cov}(X_t | \mathcal{F}_s)_{|_{t=s}} = a(X_s),$$

with $a = \sigma \sigma^* : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$.

$$(\text{cov}(X^i, X^k | \mathcal{F}_s))_{i,k=1,\dots,n}$$

Interpretation:

- $b(X_s) \equiv$ Infinitesimal drift.
- $a(X_s) \equiv$ Infinitesimal covariance.

Itô process

Definition 2

Let:

- $X : [0, \tau] \rightarrow \mathbb{R}^n$ process in L_a^2
- $\alpha \in \mathbb{R}^n$ initial condition.
- b bounded and adapted process in $\mathbb{R}^n$.
- $\{\sigma^k; k = 1, \dots, d\}$ process of $L_a^2([0, \tau]; \mathbb{R}^n)$.

We say that X is an **Itô process** if it admits a decomposition:

$$X_t = \alpha + \int_0^t b_s ds + \sum_{k=1}^d \int_0^t \sigma_s^k dW_s^k.$$

Remark:

A solution of (1) is an Itô process

$\hookrightarrow$ With $b_s = b(X_s)$ and $\sigma_s = \sigma(X_s)$.

Itô's formula for a solution of (1)

Proposition 3

Let:

- X Itô process, defined by α, b, σ .
- $f \in \mathcal{C}_b^2$.

Then $f(X_t)$ can be decomposed as:

$$\begin{aligned} f(X_t) &= f(\alpha) + \sum_{j=1}^n \int_0^t \partial_{x_j} f(X_r) b_r^j dr \\ &\quad + \sum_{j=1}^n \sum_{k=1}^d \int_0^t \partial_{x_j} f(X_r) \sigma_r^{jk} dW_r^k \\ &\quad + \frac{1}{2} \sum_{j_1, j_2=1}^n \sum_{k=1}^d \int_0^t \partial_{x_{j_1} x_{j_2}}^2 f(X_r) \sigma_r^{j_1 k} \sigma_r^{j_2 k} dr. \end{aligned}$$

Itô's formula in dimension 1

Corollary 4

Consider:

- $n = d = 1$, i.e. X and W real valued
- X Itô process, defined by α, b, σ , with $b_r, \sigma_r \in \mathbb{R}$
- $f \in \mathcal{C}_b^2(\mathbb{R})$

Then:

$$\begin{aligned} f(X_t) &= f(\alpha) + \int_0^t \partial_x f(X_r) b_r dr + \int_0^t \partial_x f(X_r) \sigma_r dW_r \\ &\quad + \frac{1}{2} \int_0^t \partial_{xx}^2 f(X_r) \underbrace{\sigma_r^2}_{=a_r} dr. \end{aligned}$$

Solution of (1): Take $b_r = b(X_r)$ and $\sigma_r = \sigma(X_r)$

Proof in a simplified setting:

$$W_t \in \mathbb{R}, \quad X_t \in \mathbb{R}, \quad \alpha \in \mathbb{R}$$

$$X_t = \alpha + \int_0^t b(x_r) dr + \int_0^t \sigma(x_r) dW_r$$

For $s=0$, we want to prove:

$$\partial_t \mathbb{E}[X_t] |_{t=0} = b(\alpha)$$

$$\partial_t V(X_t) |_{t=0} = \sigma^2(\alpha) = a(\alpha)$$

$$\mathbb{E}[M_t] = \mathbb{E}[M_0]$$

For $\mathbb{E}[X_t]$

$$= 0$$

$$\mathbb{E}[X_t] = \alpha + \int_0^t \mathbb{E}[b(x_r)] dr + \underbrace{\mathbb{E}\left[\int_0^t \sigma(x_r) dW_r\right]}_{= 0}$$

$$\Rightarrow \partial_t \mathbb{E}[X_t] = \mathbb{E}[b(x_t)]$$

$$\Rightarrow \partial_t \mathbb{E}[X_t]_{t=0} = b(\alpha)$$

Conclusion

$$\partial_t \text{Var}(X_t) |_{t=0}$$

$$= \partial_t \mathbb{E}[X_t^2] |_{t=0}$$

$$- \partial_t (\mathbb{E}[X_t])^2 |_{t=0}$$

algebra

$$= \dots =$$

$$\sigma^2(\alpha)$$

Proof of Proposition 1 for $n = d = 1$

Setting:

- X and W real valued, $a = \sigma^2$
- Shown for $s = 0 \Rightarrow$ no conditional expectation

Aim:

$$\partial_t \mathbf{E}[X_t]_{|t=0} = b(\alpha), \quad \partial_t \mathbf{Var}(X_t)_{|t=0} = \sigma^2(\alpha).$$

Method: Itô's formula in dimension 1, with $f(x) = x$ and $f(x) = x^2$

Proof of Proposition 1 for $n = d = 1$ (2)

Function $f(x) = x$: $\partial_x f = 1$, $\partial_{xx}^2 f = 0$

$\hookrightarrow$ Itô's formula in dimension 1 is the equation itself

$$X_t = \alpha + \int_0^t b(X_r) dr + \int_0^t \sigma(X_r) dW_r.$$

Expectation: Itô integral centered, hence $\mathbf{E}[X_t] = \alpha + \int_0^t \mathbf{E}[b(X_r)] dr$

$\hookrightarrow$ Differentiating at $t = 0$:

$$\partial_t \mathbf{E}[X_t]_{|_{t=0}} = \mathbf{E}[b(X_t)]_{|_{t=0}} = b(\alpha).$$

Proof of Proposition 1 for $n = d = 1$ (3)

Function $f(x) = x^2$: $\partial_x f(x) = 2x$, $\partial_{xx}^2 f = 2$

$\hookrightarrow$ Itô's formula in dimension 1 gives

$$X_t^2 = \alpha^2 + \int_0^t \left(2X_r b(X_r) + \sigma^2(X_r) \right) dr + 2 \int_0^t X_r \sigma(X_r) dW_r.$$

Expectation: Itô integral centered

$\hookrightarrow$ Differentiating at $t = 0$:

$$\partial_t \mathbf{E} \left[X_t^2 \right]_{|t=0} = 2\alpha b(\alpha) + \sigma^2(\alpha).$$

Proof of Proposition 1 for $n = d = 1$ (4)

Square of the mean: Chain rule, plus $\mathbf{E}[X_0] = \alpha$ and frame (2)

$$\partial_t (\mathbf{E}[X_t])^2|_{t=0} = 2 \mathbf{E}[X_0] \partial_t \mathbf{E}[X_t]|_{t=0} = 2\alpha b(\alpha).$$

Variance: $\mathbf{Var}(X_t) = \mathbf{E}[X_t^2] - (\mathbf{E}[X_t])^2$, so with frame (3)

$$\partial_t \mathbf{Var}(X_t)|_{t=0} = 2\alpha b(\alpha) + \sigma^2(\alpha) - 2\alpha b(\alpha) = \sigma^2(\alpha).$$

Dimension n : Same scheme, with $f(x) = x^l x^m$

Proof of Proposition 1

Simplification: This will be shown for $s = 0$
 $\hookrightarrow$ No conditional expectation to consider.

Drift term: Start from equation

$$X_t = \alpha + \int_0^t b(X_s) ds + \sum_{j=1}^d \int_0^t \sigma^j(X_s) dW_s^j.$$

The terms $\int_0^t \sigma^j(X_s) dW_s^j$ are centered. Therefore:

$$\mathbf{E}[X_t] = \alpha + \int_0^t \mathbf{E}[b(X_s)] ds.$$

and

$$\partial_t \mathbf{E}[X_t]_{|t=0} = \mathbf{E}[b(X_t)]_{|t=0} = b(\alpha).$$

Proof of Proposition 1 (2)

Coordinate product: Let

- $l, m \in \{1, \dots, n\}$.
- $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by the product $f(x) = x^l x^m$.
- $a_s = \sigma_s \sigma_s^*$.

According to Itô's formula we have:

$$\begin{aligned} X_t^l X_t^m &= \alpha^l \alpha^m + \int_0^t \left(X_r^l b_r^m + b_r^l X_r^m \right) dr \\ &\quad + \sum_{k=1}^d \int_0^t \left(X_r^l \sigma_r^{mk} + \sigma_r^{lk} X_r^m \right) dW_r^k + \sum_{k=1}^d \int_0^t a_r^{lm} dr. \end{aligned} \tag{2}$$

Proof of Proposition 1 (3)

Expected value for the product:

Taking expectation in (2) we get:

$$\partial_t \mathbf{E} [X_t^l X_t^m] = \mathbf{E} [X_t^l b_t^m + b_t^l X_t^m] + \mathbf{E} [a_t^{lm}].$$

Therefore:

$$\partial_t \mathbf{E} [X_t^l X_t^m]_{|_{t=0}} = \alpha^l b^m(\alpha) + b^l(\alpha) \alpha^m + a^{lm}(\alpha).$$

Proof of Proposition 1 (4)

Product of expected values: We have

$$\mathbf{E} [X_t^j] = \alpha^j + \int_0^t \mathbf{E} [b_s^j] ds.$$

Therefore

$$\mathbf{E} [X_t^l] \mathbf{E} [X_t^m] = \alpha^l \alpha^m + \int_0^t \mathbf{E} [b_s^l] \mathbf{E} [X_s^m] ds + \int_0^t \mathbf{E} [b_s^m] \mathbf{E} [X_s^l] ds,$$

and differentiating:

$$\partial_t \left(\mathbf{E} [X_t^l] \mathbf{E} [X_t^m] \right) \Big|_{t=0} = \alpha^l b^m(\alpha) + b^l(\alpha) \alpha^m.$$

Proof of Proposition 1 (5)

Infinitesimal covariance: With two previous identities,

$$\begin{aligned}\partial_t \mathbf{Cov} (X_t^l, X_t^m) \Big|_{t=0} &= \partial_t \left(\mathbf{E} [X_t^l X_t^m] - \mathbf{E} [X_t^l] \mathbf{E} [X_t^m] \right) \Big|_{t=0} \\ &= \partial_t \mathbf{E} [X_t^l X_t^m] \Big|_{t=0} - \partial_t \left(\mathbf{E} [X_t^l] \mathbf{E} [X_t^m] \right) \Big|_{t=0} \\ &= a^{lm}(\alpha).\end{aligned}$$

Therefore:

$$\partial_t \mathbf{Cov} (X_t) \Big|_{t=0} = a(\alpha).$$