

Itô's formula for W BM, $\mathbb{R}$ -valued

and $f: \mathbb{R} \rightarrow \mathbb{R}$
 $x \mapsto f(x)$

$$f(W_t) = \underbrace{f(0)}_{\text{Itô stochastic}} + \int_0^t \partial_x f(W_r) dW_r + \frac{1}{2} \int_0^t \partial_{xx} f(W_r) dr$$

Question : Extensions ?

Outline

- 1 Itô's integral
- 2 Itô's formula
- 3 Extensions and consequences of Itô's formula

Itô processes

Definition 14

Let:

$$W_t \in \mathbb{R}^d$$

- $X : [0, \tau] \rightarrow \mathbb{R}^n$ process of L_a^2
- $a \in \mathbb{R}^n$ initial condition
- $\{b^j; j = 1, \dots, n\}$ real bounded and adapted process
- $\{\sigma^{jk}; j = 1, \dots, n, k = 1, \dots, d\}$ process of L_a^2

We say that X is an **Itô process** if it can be decomposed as:

$$X_t^j = a^j + \underbrace{\int_0^t b_s^j ds}_{\text{drift}} + \underbrace{\int_0^t \sigma_s^{jk} dW_s^k}_{\text{stochastic}},$$

for $j = 1, \dots, n$.

Remark:

An Itô process is a particular case of **semi-martingale**.

Itô's formula for Itô processes

Theorem 15

Let:

- X Itô process, defined by a, b, σ .
- $f \in \mathcal{C}_{b,\tau}^{1,2}$.

Then $f(t, X_t)$ can be decomposed as:

$$\begin{aligned}
 f(t, X_t) &= f(0, a) + \int_0^t \partial_r f(r, X_r) dr + \sum_{j=1}^n \int_0^t \partial_{x_j} f(r, X_r) b_r^j dr \\
 &\quad + \sum_{j=1}^n \sum_{k=1}^d \int_0^t \partial_{x_j} f(r, X_r) \sigma_r^{jk} dW_r^k \\
 &\quad + \frac{1}{2} \sum_{j_1, j_2=1}^n \sum_{k=1}^d \int_0^t \partial_{x_{j_1} x_{j_2}}^2 f(r, X_r) \sigma_r^{j_1 k} \sigma_r^{j_2 k} dr.
 \end{aligned}$$

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$$\begin{aligned}
 &\sum_{k=1}^d \sigma_r^{j_1 k} \sigma_r^{j_2 k} \\
 &= (\sigma \sigma^T)^{j_1, j_2} \\
 &\equiv (a_r)^{j_1, j_2}
 \end{aligned}$$

Infinitesimal generator of Brownian motion

Proposition 16

Let:

- W a $\mathcal{F}_s$ -Wiener process in $\mathbb{R}^d$.
- $s \in \mathbb{R}_+$.
- $f \in \mathcal{C}_b^2$.

$\frac{1}{2} \Delta =$ infinitesimal generator of W

Then:

$$\lim_{h \rightarrow 0} \frac{\mathbf{E} [\delta f(W)_{s,s+h} | \mathcal{F}_s]}{h} = \frac{1}{2} \Delta f(W_s).$$

Apply Itô's formula $= \delta M_{st}$ $f \in C_b^2$

$$[f(w)]_{st} = \sum_i \int_s^t \partial_{x_i} f(w_r) dw_r + \frac{1}{2} \int_s^t \Delta f(w_r) dr$$

Here $M_t = \sum_i \int_0^t \partial_{x_i} f(w_r) dw_r$ $\nearrow$ martingale

Thus

$$\mathbb{E}[f(w)_{st} | \mathcal{F}_s] = \mathbb{E}[\delta M_{st} | \mathcal{F}_s] + \frac{1}{2} \int_s^t \mathbb{E}[\Delta f(w_r) | \mathcal{F}_s] ds$$

$= 0$

We have obtained

If $X \in \mathcal{F}$, $\mathbb{E}[X|\mathcal{F}] = X$
 $X \perp \mathcal{F}$, $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$

$$\frac{1}{h} \mathbb{E}[\Delta f(w)_{s+h} | \mathcal{F}_s] \quad \text{continuous in } r \text{ by dominated convergence}$$
$$= \frac{1}{2} \frac{1}{h} \int_s^{s+h} \mathbb{E}[\Delta f(w_r) | \mathcal{F}_s] dr$$

As $h \rightarrow 0$, we get

$$\lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E}[\Delta f(w)_{s+h} | \mathcal{F}_s]$$

$$= \frac{1}{2} \mathbb{E}[\Delta f(w_s) | \mathcal{F}_s]$$

$$= \frac{1}{2} \Delta f(w_s)$$

Proof

Recasting Itô's formula: Let

$$M_t \equiv \sum_{j=1}^d \int_0^t \partial_{x_j} f(W_r) dW_r^j.$$

Then Itô's formula can be written as:

$$\delta f(W)_{st} = \delta M_{st} + \frac{1}{2} \int_s^t \Delta f(W_r) dr.$$

Conditional expectation:

According to Proposition 9, M is a martingale. Thus:

$$\mathbf{E}[\delta f(W)_{s,s+h} | \mathcal{F}_s] = \frac{1}{2} \int_s^{s+h} \mathbf{E}[\Delta f(W_r) | \mathcal{F}_s] dr.$$

Proof (2)

Limiting procedure:

Applying dominated convergence for conditional expectation we get:

$$r \mapsto \mathbf{E} [\Delta f(W_r) | \mathcal{F}_s] \quad \text{continuous.}$$

Thus:

$$\lim_{h \rightarrow 0} \frac{1}{2h} \int_s^{s+h} \mathbf{E} [\Delta f(W_r) | \mathcal{F}_s] \, dr = \frac{1}{2} \mathbf{E} [\Delta f(W_s) | \mathcal{F}_s] = \frac{1}{2} \Delta f(W_s),$$

and

$$\lim_{h \rightarrow 0} \frac{\mathbf{E} [\delta f(W)_{s,s+h} | \mathcal{F}_s]}{h} = \frac{1}{2} \Delta f(W_s).$$

Extension for Itô processes

Theorem 17

Let:

- X Itô process, defined by a, b, σ .
- $f \in C_b^2$.

Then:

Generator for X : $b_s^j = b^j(X_s)$

$$L_s f(x) = \sum_j \partial_{x_j} f(x) b_s^j + \frac{1}{2} \sum_{j_1, j_2} \partial_{x_{j_1} x_{j_2}}^2 f(x) (\sigma \sigma^T)_s^{j_1 j_2}$$

$$\lim_{h \rightarrow 0} \frac{\mathbf{E} [\delta f(X)_{s, s+h} | \mathcal{F}_s]}{h}$$

$$= \sum_{j=1}^n \partial_{x_j} f(X_s) b_s^j + \frac{1}{2} \sum_{j_1, j_2=1}^n \sum_{k=1}^d \partial_{x_{j_1} x_{j_2}}^2 f(X_s) \sigma_s^{j_1 k} \sigma_s^{j_2 k}.$$