

Equation

$$dX_t = b(X_t)dt + \sigma(X_t)dw_t$$

Local description : letting $a = \sigma\sigma^T$,
we have

$$\partial_t \mathbb{E}[X_t | \mathcal{F}_s] |_{t=s} = b(X_s)$$

$$\partial_t \text{COV}(X_t | \mathcal{F}_s) |_{t=s} = a(X_s)$$

Today: examples

Geometrical Brownian motion

Proposition 5

Let:

- W 1-dimensional Brownian motion .
- $\mu \in \mathbb{R}$ and $\sigma > 0$

We set

$$X_t = \alpha \exp(\mu t + \sigma W_t) .$$

Then X is solution of (1) with $n = d = 1$ and:

$$b(x) = \left(\mu + \frac{\sigma^2}{2} \right) x, \quad \text{and} \quad \sigma(x) = \sigma x .$$

Proof Consider

$$\begin{aligned} X_t &= \alpha \exp(\mu t + \sigma W_t) \\ &= f(t, W_t) \end{aligned}$$

with

$$f(t, x) = \alpha \exp(\mu t + \sigma x)$$

Itô's formula

$$\begin{aligned} f(t, W_t) &= \alpha + \int_0^t \partial_t f(s, W_s) ds \\ &+ \int_0^t \partial_x f(s, W_s) dW_s + \frac{1}{2} \int_0^t \partial_{xx}^2 f(s, W_s) ds \end{aligned}$$

$$f(t,x) = \alpha \exp(\mu t + \sigma x)$$

Derivatives

$$\partial_t f(t,x) = \mu f(t,x)$$

$$\partial_x f(t,x) = \sigma f(t,x)$$

$$\partial_{xx} f(t,x) = \sigma^2 f(t,x)$$

Applying Itô

$$X_t = f(t, W_t)$$

$$= \alpha + \int_0^t \partial_t f(s, W_s) ds$$

$$+ \int_0^t \partial_x f(s, W_s) dW_s + \frac{1}{2} \int_0^t \partial_{xx}^2 f(s, W_s) ds$$

$$= \alpha + \int_0^t \mu \overbrace{f(s, W_s)}^{= X_s} ds$$

$$+ \int_0^t \sigma \overbrace{f(s, W_s)}^{X_s} dW_s + \frac{1}{2} \int_0^t \sigma^2 \overbrace{f(s, W_s)}^{X_s} ds$$

$$= \alpha + \int_0^t \left(\mu + \frac{\sigma^2}{2} \right) X_s ds + \sigma \int_0^t X_s dW_s$$

$$\Rightarrow dX_t = \left(\mu + \frac{\sigma^2}{2} \right) X_t dt + \sigma X_t dW_t$$

Geometrical Brownian motion (2)

Remarks:

- In order to show that X is solution of the equation
 $\hookrightarrow$ Apply Itô's formula.
- Exponential Brownian motion is very useful in finance (asset price):
 - ① $X_t \geq 0$.
 - ② Linear trend (infinitesimal drift): $\left(\mu + \frac{\sigma^2}{2}\right) X_t$.
 - ③ Fluctuations (infinitesimal standard deviation) proportional to X_t .

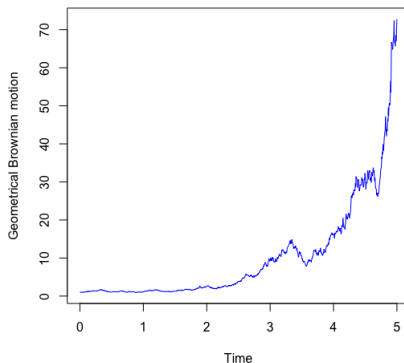
Summarized in Black and Scholes model.

Geometrical Brownian motion: illustration

Equation: Take $\mu = 1$, $\sigma = \frac{1}{2}$, $X_0 = 1$ and

$$dX_t = \left(\mu + \frac{\sigma^2}{2} \right) X_t dt + \sigma X_t dW_t$$

Simulation:



Ornstein - Uhlenbeck in $\alpha=1$.

$$dX_t = -\alpha X_t dt + \sigma dW_t$$

If W was smooth, a standard way to solve this equation (see 266-262) is based on integrating factor μ . Here we would take

$$\mu(t) = e^{+\alpha t}$$

Then we try to get an eq. for μX

Here we might have problems to write

$$d(\mu X)_t = \mu_t dX_t + X_t d\mu_t$$

$$(+ \underbrace{\langle \mu, X \rangle}_{=0 \text{ since } \mu \text{ smooth}})$$

(can be justified)

$$= \overbrace{e^{\alpha t}}^{=\mu_t} (-\alpha X_t dt + \sigma dW_t) = e^{\alpha t} \sigma dW_t + X_t (\alpha \mu_t dt)$$

Solving the equation: we have obtained
 $\mu = e^{\alpha t}$

$$d(\mu X)_t = \sigma e^{\alpha t} dW_t$$

$$\Rightarrow e^{\alpha t} X_t = \sigma \int_0^t e^{\alpha s} dW_s + \overset{X_0}{a}$$

$$\Rightarrow X_t = a e^{-\alpha t} + \underbrace{\sigma \int_0^t e^{-\alpha(t-s)} dW_s}_{\text{Wiener integral}}$$

Law of X_t $X_t \sim \mathcal{N}(m_t, \sigma_t^2)$

$$m_t = a e^{-\alpha t}$$

$$\sigma_t^2 = \sigma^2 \int_0^t e^{-2\alpha(t-s)} ds = \frac{\sigma^2 (1 - e^{-2\alpha t})}{2\alpha}$$

Recall $X_t \sim \mathcal{N}(m_t, \sigma_t^2)$

$$m_t = a e^{-\alpha t}$$

$$\sigma_t^2 = \sigma_0^2 \int_0^t e^{-2\alpha(t-s)} ds = \frac{\sigma^2 (1 - e^{-2\alpha t})}{2\alpha}$$

As $t \rightarrow \infty$

$$\lim_{t \rightarrow \infty} m_t = 0$$

$$\lim_{t \rightarrow \infty} \sigma_t^2 = \frac{\sigma^2}{2\alpha}$$

Thus $X_t \xrightarrow{(d)} \mathcal{N}(0, \frac{\sigma^2}{2\alpha})$ = $\mathcal{N}$

strong notion of convergence of X_t to its
invariant measure

Prmk we have

$$\frac{1}{T} \int_0^T \mathbb{1}_A(X_t) dt$$

$$\rightarrow \nu(A) \quad \text{a.s.}$$

(ergodic theorem)

For Markov chains we had

$$\frac{1}{N} \sum_{n=0}^N \mathbb{1}_{\{X_n=i\}} \rightarrow \nu(i)$$

Ornstein-Uhlenbeck process

Situation:

Velocity of a Brownian particle with friction α .

Equation:

$$dX_t = -\alpha X_t dt + \sigma dW_t, \quad X_0 = a \in \mathbb{R}$$

Explicit solution:

$$X_t = e^{-\alpha t} \left(a + \sigma \int_0^t e^{\alpha s} dW_s \right)$$

Distribution: For $t > 0$ we have

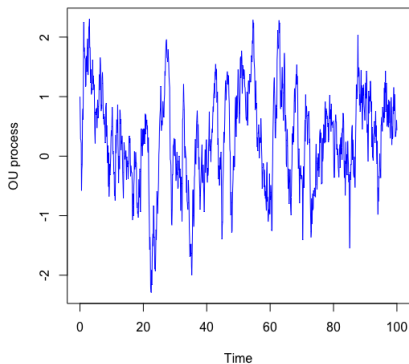
$$X_t \sim \mathcal{N} \left(a e^{-\alpha t}, \sigma_t^2 \right), \quad \text{with} \quad \sigma_t^2 = \frac{\sigma^2 (1 - e^{-2\alpha t})}{2\alpha}$$

Ornstein-Uhlenbeck: illustration

Equation: Take $\alpha = 1$ and

$$dX_t = -\alpha X_t dt + dW_t, \quad X_0 = 1$$

Simulation:



Galton-Watson process

Model: We start from $n\alpha$ persons for generation $m = 0$, then

- At each generation m , individuals have i.i.d offspring
- Common law for offspring: random variable Q .
- For $n \geq 1$, sequence of number of persons at generation m :

$$\{Z_m^n; m \geq 0\}.$$

Assumptions on Q : We suppose

- 1 $\mathbf{E}[Q] = 1 + \frac{\beta}{n}$ with $\beta > 0$.
- 2 $\mathbf{Var}(Q) = \sigma^2$ with $\sigma^2 > 0$.
- 3 For all $\delta > 0$, we have $\lim_{n \rightarrow \infty} \mathbf{E}[Q^2 \mathbf{1}_{(Q > \delta n)}] = 0$

Feller diffusion

Expectation and variance computations: Thanks to offspring $\perp\!\!\!\perp$, we have

$$\mathbf{E}[Z_1^n | Z_0^n = n\alpha] = n\alpha \left(1 + \frac{\beta}{n}\right), \quad \mathbf{Var}(Z_1^n | Z_0^n = n\alpha) = n\alpha\sigma^2.$$

Scaling: We set $X_t^n = \frac{1}{n}Z_{[nt]}^n$. Then:

$$\mathbf{E}[X_{1/n}^n - \alpha | X_0^n = \alpha] = \alpha\beta \frac{1}{n}, \quad \mathbf{Var}(X_{1/n}^n | X_0^n = \alpha) = \alpha\sigma^2 \frac{1}{n}.$$

Limiting equation: Computing limiting drift and variance, we get

$$dX_t = \beta X_t dt + \sigma \sqrt{X_t} dW_t, \quad X_0 = \alpha.$$

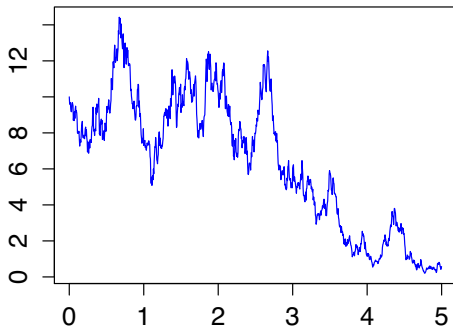
One can show that $\lim_{n \rightarrow \infty} X^n = X$ in law.

Feller diffusion: illustration

Equation:

$$dX_t = .02X_t dt + 2\sqrt{X_t} dW_t$$

Simulation:



Outline

1 Introduction and examples

2 Existence and uniqueness

- A quick contraction argument
- Picard iteration on the whole interval

Definition of solution

$$(1) \quad dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

Definition 6

We say that $(X, W, (\mathcal{F}_t)_{t \geq 0})$ is solution of (1) if:

① W is a $\mathcal{F}_t$ -Brownian motion.

$$\text{Hyp: } \sigma^i(x) \in \mathcal{L}_a^2$$

② X satisfies $X_t = \alpha + \int_0^t b(X_s) ds + \sum_{j=1}^d \int_0^t \sigma^j(X_s) dW_s^j$ for $t \geq 0$.

Definition 7

We say that (1) admits a **strong solution** if:

$\hookrightarrow$ One can take $\mathcal{F}_t = \mathcal{F}_t^W$ in Definition 6.

no more information in X than what
is contained in W

$$\mathcal{F}_t^W = \sigma(W_s; s \in [0, t])$$

Pathwise uniqueness

Definition 8

Pathwise uniqueness: If X^1, X^2 are two solutions of (1) with:

① $X_0^1 = X_0^2 = \alpha$.

② Same Brownian motion W .

Then:

$$\mathbf{P} \left(X_t^1 = X_t^2 \text{ for all } t \geq 0 \right) = 1.$$
 x^1, x^2 indistinguishable, much stronger than x^2 modification of x^1

Remark: Absence of strong solution and non pathwise uniqueness

$\hookrightarrow$ For very irregular coefficients b, σ .

Existence and uniqueness

Theorem 9

We assume that b and σ are Lipschitz functions:

There exist $c_\sigma, c_b > 0$ such that for all $x, y \in \mathbb{R}^n$ we have

$$|\sigma^j(x) - \sigma^j(y)| \leq c_\sigma |x - y|, \quad |b(x) - b(y)| \leq c_b |x - y|.$$

Then on every interval $[0, \tau]$, equation (1) admits:

- ① *A strong solution in $L^2(\Omega; \mathcal{C}([0, \tau]))$.*
- ② *Pathwise uniqueness in $L^2(\Omega; \mathcal{C}([0, \tau]))$.*

