

Existence and uniqueness

Theorem 9

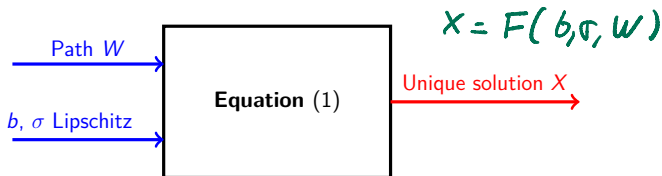
We assume that b and σ are Lipschitz functions:

There exist $c_\sigma, c_b > 0$ such that for all $x, y \in \mathbb{R}^n$ we have

$$|\sigma^j(x) - \sigma^j(y)| \leq c_\sigma |x - y|, \quad |b(x) - b(y)| \leq c_b |x - y|.$$

Then on every interval $[0, \tau]$, equation (1) admits:

- ① *A strong solution in $L^2(\Omega; \mathcal{C}([0, \tau]))$.*
- ② *Pathwise uniqueness in $L^2(\Omega; \mathcal{C}([0, \tau]))$.*



State space for X

$$\mathcal{H}_T = L^2(\Omega; C([0, T]))$$

here we take 1-d case

equipped with

$$\|z\|_{\mathcal{H}_T}^2 = \mathbb{E} \left[\sup_{t \leq T} |z_t|^2 \right]$$

Map Γ Take $z \in \mathcal{H}_T$. Define

$\tilde{z} = \Gamma(z)$ by setting

$$\tilde{z}_t = \alpha + \int_0^t b(\tilde{z}_s) ds + \int_0^t \sigma(\tilde{z}_s) dW_s$$

Strategy: Banach fixed point. If we have

$$\|P(z) - P(y)\|_{\mathcal{H}_c} \leq c \|z - y\|_{\mathcal{H}_c},$$

for $c \in [0, 1)$, then there

exists a unique solution in $\mathcal{H}_c$

$$\text{to } Y = P(Y)$$

$$\Leftrightarrow Y_t = \alpha + \int_0^t b(Y_s) ds + \int_0^t \sigma(Y_s) dW_s$$

$\hookrightarrow$ unique sol. in $\mathcal{H}_c$ to the equation

Outline

1 Introduction and examples

2 Existence and uniqueness

- A quick contraction argument
- Picard iteration on the whole interval

Quick contraction argument (1)

Space: $\mathcal{H}_\tau = L^2(\Omega; \mathcal{C}([0, \tau]; \mathbb{R}^n))$, adapted processes, with norm

$$\|Z\|_{\mathcal{H}_\tau}^2 = \mathbf{E} \left[\sup_{t \leq \tau} |Z_t|^2 \right].$$

Picard map: $\Gamma : \mathcal{H}_\tau \rightarrow \mathcal{H}_\tau$ given by $\Gamma(Y) \equiv \tilde{Y}$, with

$$\tilde{Y}_t = \alpha + \int_0^t b(Y_s) ds + \sum_{j=1}^d \int_0^t \sigma^j(Y_s) dW_s^j.$$

$\hookrightarrow X$ solves (1) iff $X = \Gamma(X)$

Bounds on application Γ

$$\|Y\|_{L_a^2}^2 = \int_0^\tau \mathbb{E}[|Y_r|^2] dr$$

Lemma 10

Let:

- $Y, Z \in L_a^2([0, \tau])$.
- $\tilde{Y} = \Gamma(Y)$, $\tilde{Z} = \Gamma(Z)$.

We assume that $c_\sigma, c_b \leq K$. Then:

$$\mathbb{E} \left[\sup_{t \leq \tau} |\tilde{Y}_t - \tilde{Z}_t|^2 \right] \leq c_{K, \tau, d} \mathbb{E} \left[\int_0^\tau |Y_t - Z_t|^2 dt \right].$$

with:

$$c_{K, \tau, d} = (2\tau d + 8d^2) K^2.$$

Eq for $\tilde{Y} - \tilde{Z}$. We have

$$\tilde{Y}_t = \alpha + \int_0^t b(Y_s) ds + \int_0^t \sigma(Y_s) dW_s$$

$$\tilde{Z}_t = \alpha + \int_0^t b(Z_s) ds + \int_0^t \sigma(Z_s) dW_s$$

$\Rightarrow$

$$\begin{aligned} \tilde{Y}_t - \tilde{Z}_t &= \int_0^t [b(Y_s) - b(Z_s)] ds \quad \textcolor{blue}{) \equiv J_{t,1}} \\ &\quad + \int_0^t [\sigma(Y_s) - \sigma(Z_s)] dW_s \quad \textcolor{green}{) \equiv J_{t,2}} \end{aligned}$$

$$\begin{aligned}\tilde{Y}_t - \tilde{Z}_t &= \int_0^t [b(Y_s) - b(Z_s)] ds \equiv J_{t,1} \\ &+ \int_0^t [\sigma(Y_s) - \sigma(Z_s)] dW_s \equiv J_{t,2}\end{aligned}$$

We get

$$\begin{aligned}|\tilde{Y}_t - \tilde{Z}_t|^2 &= |J_{t,1} + J_{t,2}|^2 \\ &\leq 2 (|J_{t,1}|^2 + |J_{t,2}|^2)\end{aligned}$$

$$\Rightarrow \sup_{t \leq c} |\tilde{Y}_t - \tilde{Z}_t|^2 \leq 2 \left(\sup_{t \leq c} J_{t,1}^2 + \sup_{t \leq c} J_{t,2}^2 \right)$$

$$\begin{aligned} \tilde{Y}_t - \tilde{Z}_t &= \int_0^t [b(Y_s) - b(Z_s)] ds \equiv J_{t,1} \\ &+ \int_0^t [\sigma(Y_s) - \sigma(Z_s)] dW_s \equiv J_{t,2} \end{aligned}$$

$$\sup_{t \leq T} J_{t,1}^2 = t^2 \left(\int_0^t b(Y_s) - b(Z_s) \frac{ds}{t} \right)^2$$

Jensen

$$\leq t^2 \int_0^t |b(Y_s) - b(Z_s)|^2 \frac{ds}{t}$$

Lipschitz

$$\leq t L_b^2 \int_0^t |Y_s - Z_s|^2 ds$$

$$\leq T L_b^2 \int_0^T |Y_s - Z_s|^2 ds$$

$$\int_0^t [\sigma(Y_s) - \sigma(Z_s)] dW_s \equiv J_{t,Z}$$

Consider $t \mapsto J_{t,Z}$ as a martingale M
 In case $M = W$, we have seen

$$\underbrace{\sup_{t \leq Z} W_t}_{\nearrow \text{with } Z} \stackrel{(d)}{=} \overbrace{|W_Z|}^{\text{not } \nearrow} \quad (\text{reflection})$$

$\hookrightarrow$ true for Z fixed
 but not as processes

For a general martingale, Doob's ineq:

Take $M_t = \int_0^t u_s dw_s$, $u \in L_a^2$.

Then

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq z} M_t^2 \right] &\leq 4 \mathbb{E} [|M_z|^2] \\ &= 4 \int_0^z \mathbb{E} [|u_r|^2] dr \end{aligned}$$

$$\int_0^t (\sigma(Y_s) - \sigma(Z_s)) dW_s \equiv J_{t,2}$$

Bounding $J_{t,2}$

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \leq T} |J_{t,2}|^2 \right] \\ \text{Doob's} & \leq 4 \int_0^T |\sigma(Y_s) - \sigma(Z_s)|^2 ds \\ & \leq 4 L_\sigma^2 \int_0^T |Y_s - Z_s|^2 ds \end{aligned}$$

Summarizing

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \leq z} |\tilde{Y}_t - \tilde{Z}_t|^2 \right] \\ & \leq \mathbb{E} \left[\sup_{t \leq z} |\tilde{X}_{t,1}|^2 \right] + \mathbb{E} \left[\sup_{t \leq z} |\tilde{D}_{t,z}|^2 \right] \\ & \leq C_{b,\sigma,z} \mathbb{E} \left\{ \int_0^z |Y_s - Z_s|^2 ds \right\} \end{aligned}$$

↪ proves the lemma

Conclusion We have seen

$$\begin{aligned} \|\tilde{Y} - \tilde{Z}\|_{\mathcal{H}_T}^2 &\equiv \mathbb{E} \left[\sup_{t \leq T} |Y_t - Z_t|^2 \right] \leq \sup_{t \leq T} |\tilde{Y}_t - \tilde{Z}_t|^2 \\ &\leq C_{b, \sigma, T} \mathbb{E} \left\{ \int_0^T |Y_t - Z_t|^2 dt \right\} \\ &\leq C_{b, \sigma, T} \mathbb{E} \left[\sup_{t \leq T} |Y_t - Z_t|^2 \right] \int_0^T dt \\ &\leq C_{b, \sigma, T} T \|\tilde{Y} - \tilde{Z}\|_{\mathcal{H}_T}^2 \end{aligned}$$

Contraction: as soon as

$$C_{b, \sigma, T} T < 1 \rightarrow \text{Existence-uniq. on } [0, T]$$

Proof for $n = d = 1$

Expression for the difference: We have

$$\begin{aligned}\tilde{Y}_t - \tilde{Z}_t &= \int_0^t [b(Y_s) - b(Z_s)] ds + \int_0^t [\sigma(Y_s) - \sigma(Z_s)] dW_s \\ &\equiv J_{t,1} + J_{t,2}.\end{aligned}$$

Therefore

$$|\tilde{Y}_t - \tilde{Z}_t|^2 \leq 2(|J_{t,1}|^2 + |J_{t,2}|^2)$$

and:

$$\sup_{t \leq \tau} |\tilde{Y}_t - \tilde{Z}_t|^2 \leq 2 \left(\sup_{t \leq \tau} |J_{t,1}|^2 + \sup_{t \leq \tau} |J_{t,2}|^2 \right).$$

Bounds for the Lebesgue integral

Application of Jensen: We have

$$\begin{aligned}|J_{t,1}|^2 &= \left| \int_0^t [b(Y_s) - b(Z_s)] ds \right|^2 \\ &\leq t \int_0^t |b(Y_s) - b(Z_s)|^2 ds \\ &\leq \tau \int_0^\tau |b(Y_s) - b(Z_s)|^2 ds.\end{aligned}$$

Lipschitz property for b : We get

$$\begin{aligned}\sup_{t \leq \tau} |J_{t,1}|^2 &\leq \tau K^2 \int_0^\tau |Y_s - Z_s|^2 ds \\ \mathbf{E} \left[\sup_{t \leq \tau} |J_{t,1}|^2 \right] &\leq \tau K^2 \int_0^\tau \mathbf{E} [|Y_s - Z_s|^2] ds\end{aligned}$$

Doob's maximal inequality

Proposition 11

Let $\tau > 0$ and:

- W standard Brownian motion.
- $u \in L_a^2([0, \tau])$.
- $M_t \equiv \int_0^t u_r dW_r$.

Then we have:

$$\mathbf{E} \left[\sup_{t \in [0, \tau]} |M_t|^2 \right] \leq 4 \mathbf{E} \left[|M_\tau|^2 \right] = 4 \mathbf{E} \left[\int_0^\tau u_r^2 dr \right].$$

Bounds for the stochastic integral

Recall: $J_{t,2} = \int_0^t [\sigma(Y_s) - \sigma(Z_s)] dW_s$

Application of Proposition 11: We have

$$\begin{aligned} \mathbf{E} \left[\sup_{t \leq \tau} |J_{t,2}|^2 \right] &= \mathbf{E} \left[\sup_{t \leq \tau} \left| \int_0^t [\sigma(Y_s) - \sigma(Z_s)] dW_s \right|^2 \right] \\ &\leq 4 \mathbf{E} \left[\int_0^\tau |\sigma(Y_s) - \sigma(Z_s)|^2 dr \right] \end{aligned}$$

Conclusion: Lemma 10 is shown

↪ Combining inequalities for $J_{t,1}$ and $J_{t,2}$.

Quick contraction argument (2)

Recall (bound on Γ): For $Y, Z \in \mathcal{H}_\tau$,

$$\mathbf{E} \left[\sup_{t \leq \tau} |\tilde{Y}_t - \tilde{Z}_t|^2 \right] \leq c_{K, \tau, d} \mathbf{E} \left[\int_0^\tau |Y_r - Z_r|^2 dr \right].$$

Sup inside the integral: We have

$$\int_0^\tau |Y_r - Z_r|^2 dr \leq \tau \sup_{r \leq \tau} |Y_r - Z_r|^2$$

therefore

$$\|\Gamma(Y) - \Gamma(Z)\|_{\mathcal{H}_\tau}^2 \leq c_{K, \tau, d} \tau \|Y - Z\|_{\mathcal{H}_\tau}^2.$$

Quick contraction argument (3)

Contraction: $c_{K,\tau,d} \tau \rightarrow 0$ as $\tau \rightarrow 0$

$\hookrightarrow \Gamma$ is a contraction on $\mathcal{H}_\tau$ for τ small enough

Fixed point: Unique $X \in \mathcal{H}_\tau$ with $X = \Gamma(X)$

$\hookrightarrow$ Solution of (1) on the small interval $[0, \tau]$

Left aside:

- Completeness of $\mathcal{H}_\tau$, needed for the fixed point theorem
- Patching small intervals to reach an arbitrary horizon