Outline

Stochastic processes

Definition 1

Let:

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space.
- $I \subset \mathbb{R}_+$ interval.
- $\{X_t;\ t \in I\}$ family of random variables, $\mathbb{R}^n$-valued

Then:

1. If $\omega \mapsto X_t(\omega)$ measurable, X is a stochastic process
2. $t \mapsto X_t(\omega)$ is called a path
3. X is continuous if its paths are continuous a.s.

$\hookrightarrow \quad \mathbb{P}(\ t \mapsto X_t \ \text{continuous}) = 1$

$(\Omega, \mathcal{F}, P)$ probability space means

(1) $\mathcal{F}$ is a σ-algebra on Ω

(a) $\emptyset \in \mathcal{F}$ $\Omega \in \mathcal{F}$

(b) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(c) $\{A_n ; n \geq 1\}$ sequence of sets s.t. $A_n \in \mathcal{F}$, then

$$\bigcup_{n \geq 1} A_n \in \mathcal{F}$$

Rmk Ω is an abstract set

More concrete examples:

(i) Roll a dice

$$\Omega = \{1, 2, \ldots, 6\}$$

$$\mathcal{F} = \mathcal{P}(\Omega) \quad (\text{every subset of } \Omega)$$

(ii) Gaussian in $\mathbb{R}$

Then $\Omega = \mathbb{R}$

$$\mathcal{F} = \mathcal{B}(\mathbb{R})$$

(iii) Brownian motion

$\Omega = C(\mathbb{R}_+)$ (continuous functions)

$\mathcal{F} = \ ?$

(2) $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$:

(a) $\mathbb{P}(\Omega) = 1$ (b) $\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$

(c) $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$
if the A_n's are disjoint

Z is a random variable means:

$$Z : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$$

measurable

meaning: $\forall A \in \mathcal{B}(\mathbb{R}^n)$, we have $\boxed{X^{-1}(A) \in \mathcal{F}}$

Rmk In probability, when we write

$$\boxed{\mathbb{P}(X \in A)}$$

it means

$$\boxed{\mathbb{P}(\{\omega \in \Omega \,;\, X(\omega) \in A\})}$$

Modifications of processes

Definition 2

Let X and Y be two processes on $(\Omega, \mathcal{F}, \mathbf{P})$.

1. X is a modification of Y if

$$\mathbf{P}\left(X_t = Y_t\right) = 1, \quad \text{for all} \quad t \in I$$

2. X and Y are non-distinguishable if

$$\mathbf{P}\left(X_t = Y_t \text{ for all } t \in I\right) = 1$$

Remarks:

(i) Relation (2) implicitly means that $(X_t = Y_t \text{ for all } t \in I) \in \mathcal{F}$

(ii) (2) is much stronger than (1)

(iii) If X and Y are continuous, (2) $\Longleftrightarrow$ (1)

Rmk For $t \in I$, set

$$A_t = (X_t = Y_t)$$

Since $\omega \mapsto X_t(\omega)$ and $\omega \mapsto Y_t(\omega)$ are measurable, we have
$A_t \in \mathcal{F}$

Modification: X modification of Y if

$$\boxed{\mathbb{P}(A_t) = 1 \quad \forall\, t \in I}$$

Non distinguishable : when

$$\boxed{\mathbb{P}\left(\bigcap_{t\in I} A_t\right) = 1}$$

Pb: $A_t \in \mathcal{F} \Rightarrow \bigcap_{n=1}^{\infty} A_{t_n} \in \mathcal{F}$

However, for the uncountable

$\bigcap_{t\in I} A_t$, we are not sure

Rmk 2 If we know that both X & Y are continuous, modification $\Leftrightarrow$ indistinguishable

Filtrations

Filtration: Increasing sequence of σ-algebras, i.e
$\hookrightarrow$ If $s < t$, then $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$.

Interpretation: $\mathcal{F}_t$ summarizes an information obtained at time t

Negligible sets: $\mathcal{N} = \{F \in \mathcal{F};\, \mathbf{P}(F) = 0\}$ enough: $\mathcal{N} \subset \mathcal{F}_0$

Complete filtration: Whenever $\mathcal{N} \subset \mathcal{F}_t$ for all $t \in I$

Stochastic basis: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbf{P})$ with a complete $(\mathcal{F}_t)_{t\in I}$

Remark: Filtration $(\mathcal{F}_t)_{t\in I}$ can always be thought of as complete
$\hookrightarrow$ One replaces $\mathcal{F}_t$ by $\hat{\mathcal{F}}_t = \sigma(\mathcal{F}_t, \mathcal{N})$

Adaptation

Definition 3

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbf{P})$ stochastic basis
- $\{X_t;\ t \in I\}$ stochastic process

We say that X is $\mathcal{F}_t$-adapted if for all $t \in I$:

$$X_t : (\Omega, \mathcal{F}_t) \longrightarrow (\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m)) \quad \text{is measurable}$$

Remarks:

(i) Let $\mathcal{F}_t^X = \sigma\{X_s;\ s \leq t\}$ the natural filtration of X.
$\hookrightarrow$ Process X is always $\mathcal{F}_t^X$-adapted.

(ii) A process X is $\mathcal{F}_t$-adapted iff $\mathcal{F}_t^X \subset \mathcal{F}_t$

awful notation

Recall: X $\mathcal{F}_t$-adapted if X_t is $\mathcal{F}_t$-measurable $(X_t \in \mathcal{F}_t)$

Finance interpretation:

If X_t = strategy for a portfolio then it is natural to assume that X_t depends only on your information up to time t

↳ $\mathcal{F}_t$

In discrete time, for a strategy, we impose

$$X_n \in \mathcal{F}_{n-1} \qquad \text{(predictable)}$$

In continuous time we might impose

$$\boxed{X_t \in \mathcal{F}_{t-}} = \bigcup_{s<t} \mathcal{F}_s$$

Another remark: If

$$\mathcal{F}_t^X = \underbrace{\sigma(X_s;\ s \leq t)}_{\hookrightarrow \text{ natural filtration}},$$

then

$$X_t \in \mathcal{F}_t^X$$

More generally

$$\boxed{X_t \text{ adapted if } \mathcal{F}_t^X \subset \mathcal{F}_t \quad \forall t \in I}$$

One possible use of complete $\mathcal{F}_t$

Consider X adapted ($X_t \in \mathcal{F}_t$)

Y modification of X $\hookrightarrow \mathbb{P}(A_t) = 1$

Is Y adapted? $(X_t = Y_t)$

We define $\tilde{Y}_t$ as:

$$\tilde{Y}_t(\omega) = \begin{cases} X_t(\omega) & \text{if } \omega \in A_t \\ 0 & \text{if } \omega \in A_t^c \end{cases}$$

Then $\tilde{Y}_t \in \mathcal{F}_t$, since $A_t^c \in \mathcal{N} \subset \mathcal{F}_t$

Outline

Definition of the Wiener process ≡ Brownian motion

Notation: For a function f, $\delta f_{st} \equiv f_t - f_s$

Definition 4

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space $\mathcal{F}_t$
- $\{W_t;\ t \geq 0\}$ stochastic process, $\mathbb{R}$-valued

can be deduced from 1-2-3 (for a modification of W)

We say that W is a Wiener process if:

1. $W_0 = 0$ almost surely
2. Let $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n$. The increments

$$\delta W_{t_0 t_1}, \delta W_{t_1 t_2}, \ldots, \delta W_{t_{n-1} t_n} \quad \text{are independent}$$

$W_{t_n} - W_{t_{n-1}}$

3. For $0 \leq s < t$ we have $\delta W_{st} \sim \mathcal{N}(0, t - s)$
4. W has continuous paths almost surely

(5) $W_t \in \mathcal{F}_t$

In particular, if W is a WP,

$$W_t \sim \mathcal{N}(0, t)$$

If $Z \sim \mathcal{N}(\mu, \sigma^2)$, its density is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Density for W_t: $\boxed{p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}}$

Illustration: chaotic path

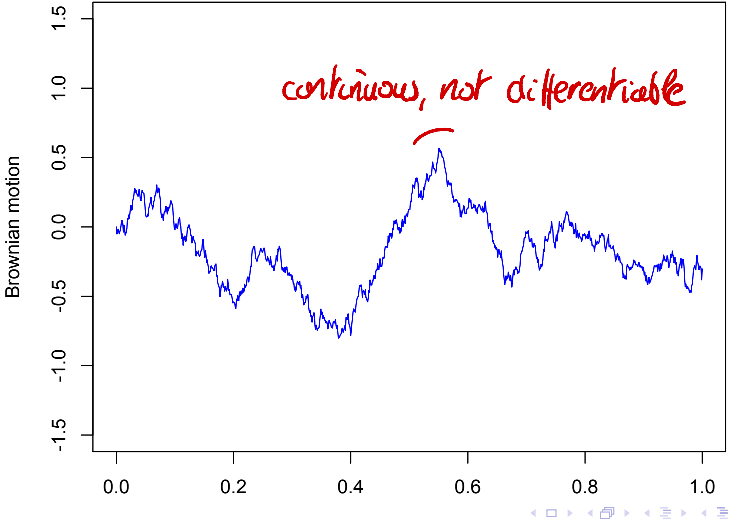
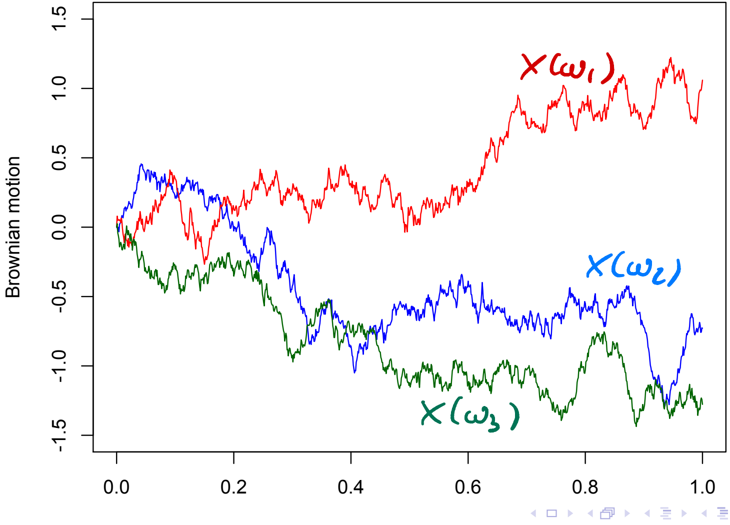


Illustration: random path



Existence of the Wiener process

Theorem 5

There exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ on which one can construct a Wiener process.

Classical constructions:

- Kolmogorov's extension theorem
- Limit of a renormalized random walk
- Lévy-Ciesilski's construction

Kolmogorov's approach

If we are given "compatible" ~~finite dimensional distributions~~ f.d.d.

for any random vector

$$(W_{t_1}, W_{t_2}, \ldots, W_{t_n})$$

for any $t_1 < t_2 \cdots < t_n$,

$$\boxed{\text{then } \exists \text{ a stochastic process } W \text{ with those f.d.d}}$$

Here we know the f.d.d's

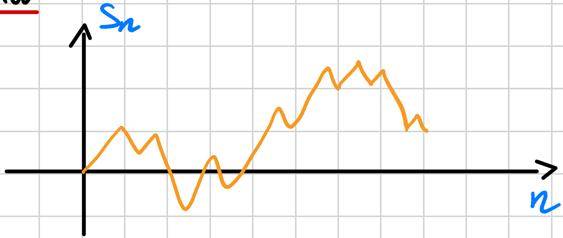
$$(W_{t_1}, W_{t_2}, \dots, W_{t_n})$$

$$\hookrightarrow (W_{t_1}, \delta W_{t_1 t_2}, \dots, \delta W_{t_{n-1} t_n})$$

are $\perp\!\!\!\perp$, $\delta W_{t_j t_{j+1}} \sim \mathcal{N}(0, t_{j+1} - t_j)$

Thus $\boxed{\text{we can apply Kolmogorov}}$

RW



Donsker:

$$\frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor} \text{ "} \xrightarrow{(d)} \text{" } W_t$$