Existence of the Wiener process

Theorem 5

There exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ on which one can construct a Wiener process.

Classical constructions:

- Kolmogorov's extension theorem
- Limit of a renormalized random walk
- Lévy-Ciesilski's construction

Wiener process W:

$dW_t \approx \sqrt{t}\, dt$

(i) $W_0 = 0$

(ii) $\delta W_{st} \sim \mathcal{N}(0, t-s)$

(iii) $\delta W_{0t_1}, \delta W_{t_1 t_2}, \ldots, \delta W_{t_{n-1} t_n}$

are $\perp\!\!\!\perp$

(iv) W continuous

Haar functions

Rmk: Haar functions are wavelet type functions
↳ take 1 shape, then scale & shift

Definition 6

We define a family of functions $\{h_k : [0,1] \to \mathbb{R};\ k \geq 0\}$:

$$
\begin{aligned}
h_0(t) &= \mathbf{1} \\
h_1(t) &= \mathbf{1}_{[0,1/2]}(t) - \mathbf{1}_{(1/2,1]}(t),
\end{aligned}
$$

and for $n \geq 1$ and $2^n \leq k < 2^{n+1}$:

$$
h_k(t) = 2^{n/2}\,\mathbf{1}_{\left[\frac{k-2^n}{2^n}, \frac{k-2^n+1/2}{2^n}\right]}(t) - 2^{n/2}\,\mathbf{1}_{\left(\frac{k-2^n+1/2}{2^n}, \frac{k-2^n+1}{2^n}\right]}(t)
$$

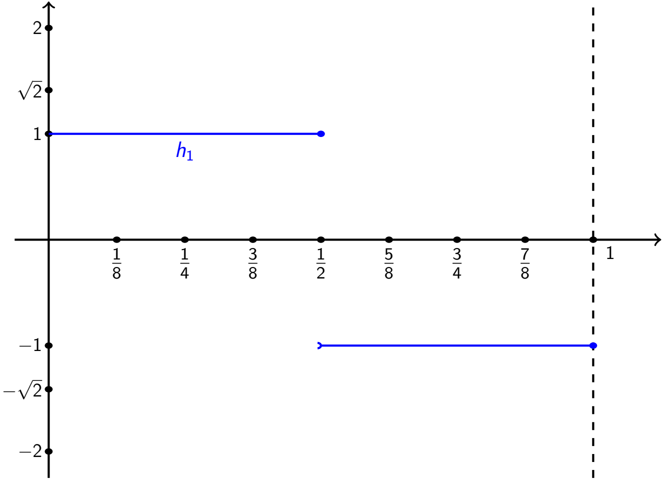
Lemma 7

The functions $\{h_k : [0,1] \to \mathbb{R};\ k \geq 0\}$ form an orthonormal basis of $L^2([0,1])$.

Haar functions: illustration

$h_0 = \mathbb{1}$

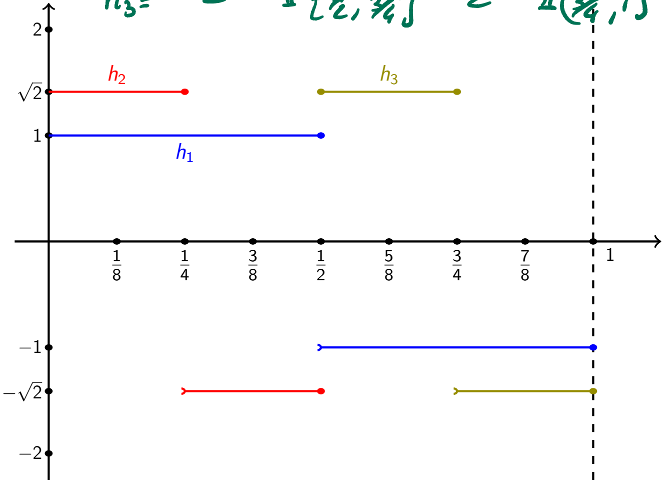
$h_1(t) = \mathbb{1}_{[0,1/2]}(t) - \mathbb{1}_{(1/2,1]}(t)$



Haar functions: illustration

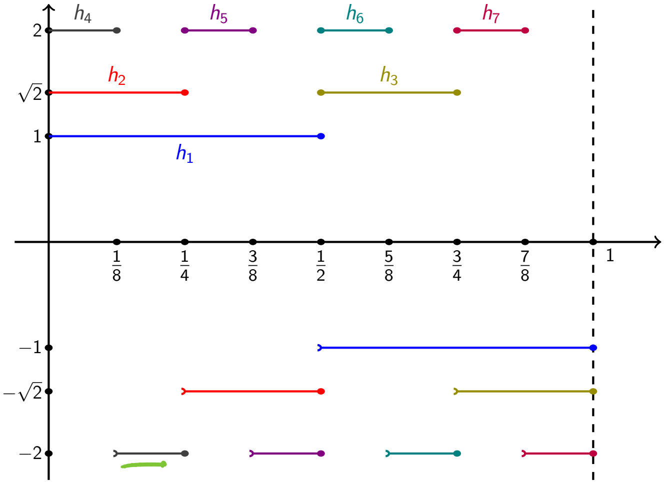
$h_2 = 2^{\frac{1}{2}} \mathbb{1}_{[0,\frac{1}{4}]} - 2^{\frac{1}{2}} \mathbb{1}_{(\frac{1}{4},\frac{1}{2}]}$

$h_3 = 2^{\frac{1}{2}} \mathbb{1}_{[\frac{1}{2},\frac{3}{4}]} - 2^{\frac{1}{2}} \mathbb{1}_{(\frac{3}{4},1]}$



Haar functions: illustration

$$\int_0^1 h_2(t)\, h_4(t) = \sqrt{2} \int_0^1 h_4(t)\, dt = 0$$



$$h_k(t) = 2^{n/2}\, \mathbb{1}_{\left[\frac{k-2^n}{2^n}, \frac{k-2^n+1/2}{2^n}\right]}(t)$$ $k = 2^n, \dots, 2^{n+1}-1$

$$- 2^{n/2}\, \mathbb{1}_{\left(\frac{k-2^n+1/2}{2^n}, \frac{k-2^n+1}{2^n}\right]}(t)$$

Call $D_k = \left[\frac{k-2^n}{2^n}, \frac{k-2^n+1/2}{2^n}\right] \cup \left(\frac{k-2^n+1/2}{2^n}, \frac{k-2^n+1}{2^n}\right]$

Then $|D_k| = \frac{1}{2^n}$

On $[0,1]$ the measure is λ = Lebesgue

Thus $\int_{D_k} h_k^2(t)\, dt$

$= (2^{n/2})^2 \int_{D_k} dt$

$= \frac{2^n}{2^n} = 1$

$$\Rightarrow \quad \boxed{\|h_k\|_{L^2([0,1])} = 1}$$

Claim 2: $\langle h_k, h_\ell \rangle_{L^2([0,1])}$

$$= \int_0^1 h_k(t)\, h_\ell(t)\, dt = \mathbb{1}_{\{k=\ell\}}$$

Hyp: $k < \ell$. Then we have only 2 possibilities

(i) $\mathrm{Supp}(h_\ell) \subset \mathrm{Supp}(h_k)$

(ii) $\mathrm{Supp}(h_\ell) \cap \mathrm{Supp}(h_k) = \emptyset$

If $f : [0,1] \to \mathbb{R}$, $\mathrm{Supp}(f) = \overline{\{x \in [0,1];\ f(x) \neq 0\}}$

(i) $\operatorname{Supp}(h_\ell) \subset \operatorname{Supp}(h_k)$

$\ell > k$

$k \in \{2^n, \dots, 2^{n+1}-1\}$

$$\Rightarrow \langle h_k, h_\ell \rangle = \pm 2^{-n/2} \int h_\ell(t)\, dt = 0$$

(ii) $\operatorname{Supp}(h_\ell) \cap \operatorname{Supp}(h_k) = \emptyset$

$$\Rightarrow \int_0^1 \underbrace{h_k(t)\, h_\ell(t)}_{=0}\, dt = 0$$

<u>Conclusion:</u> If $k < \ell$, $\langle h_k, h_\ell \rangle = 0$

If $k = \ell$, $\langle h_k, h_k \rangle = \|h_k\|^2 = 1$

Claim 3 The system is complete.

I.e. if $f \in L^2([0,1])$

s.t. $\langle f, h_k \rangle = 0 \quad \forall k \geq 0$

Then $f = 0$

$h_1 = \mathbb{1}_{[0,\frac{1}{2}]} - \mathbb{1}_{(\frac{1}{2},1]}$

Here take $f \in L^2([0,1])$ s.t.

$$\langle f, h_k \rangle = 0 \quad \forall k \geq 0$$

Then

$$\langle f, h_0 \rangle = 0 \Rightarrow \overbrace{\int_0^1 f(t)\,dt}^{\int_0^{\frac{1}{2}} f(t)\,dt + \int_{\frac{1}{2}}^1 f(t)\,dt} = 0$$

$$\langle f, h_1 \rangle = 0 \Rightarrow \int_0^{\frac{1}{2}} f(t)\,dt - \int_{\frac{1}{2}}^1 f(t)\,dt = 0$$

We have found that

$$a = \int_0^{\frac{1}{2}} f(t)\,dt \quad , \quad b = \int_{\frac{1}{2}}^{1} f(t)\,dt$$

solve

$$\begin{cases} a + b = 0 \\ a - b = 0 \end{cases}$$

Thus $\boxed{a = 0 \quad , \quad b = 0}$

In the same way, we find that

$$\int_{t_1}^{t_2} f(s)\, dt = 0$$

for dyadic t_1, t_2

$t_1 = \frac{j}{2^n}$ for $j, n \in \mathbb{N}$

Set $F(t) = \int_0^t f(s)\, ds$. Then

(i) $F(t_2) - F(t_1) = 0 \quad \forall\, t_1, t_2$ dyadic

(ii) F continuous

$\Rightarrow F = 0$

Conclusion

$$F(t) = \int_0^t f(s)\,ds = 0 \quad \forall t \in [0,1]$$

$$\Rightarrow \quad \boxed{f = 0} \quad \text{a.e on } [0,1]$$

(Lebesgue's differentiation thm)

$$\lambda(\{x \in [0,1] ;\ f(x) \neq 0\}) = 0$$

Proof

Norm: For $2^n \leq k < 2^{n+1}$, we have

$$\int_0^1 h_k^2(t)\,dt = 2^n \int_{\frac{k-2^n}{2^n}}^{\frac{k-2^n+1}{2^n}} dt = 1.$$

Orthogonality: If $k < l$, we have two situations:

(i) $\mathrm{Supp}(h_k) \cap \mathrm{Supp}(h_l) = \varnothing$.
Then trivially $\langle h_k,\, h_l \rangle_{L^2([0,1])} = 0$

(ii) $\mathrm{Supp}(h_l) \subset \mathrm{Supp}(h_k)$.
Then if $2^n \leq k < 2^{n+1}$ we have:

$$\langle h_k,\, h_l \rangle_{L^2([0,1])} = \pm 2^{n/2} \int_0^1 h_l(t)\,dt = 0.$$

Proof (2)

Complete system: Let $f \in L^2([0,1])$ s.t $\langle f,\, h_k\rangle = 0$ for all k.
$\hookrightarrow$ We will show that $f = 0$ almost everywhere.

Step 1: Analyzing the relations $\langle f,\, h_k\rangle = 0$
$\hookrightarrow$ We show that $\int_s^t f(u)\, du = 0$ for dyadic r, s.

Step 2: Since $\int_s^t f(u)\, du = 0$ for dyadic r, s, we have

$$f(t) = \partial_t \left(\int_0^t f(u)\, du \right) = 0, \quad \text{almost everywhere,}$$

according to Lebesgue's derivation theorem.

Schauder functions

Definition 8

We define a family of functions $\{s_k : [0,1] \to \mathbb{R};\ k \geq 0\}$:

$$s_k(t) = \int_0^t h_k(u)\, du$$

Lemma 9

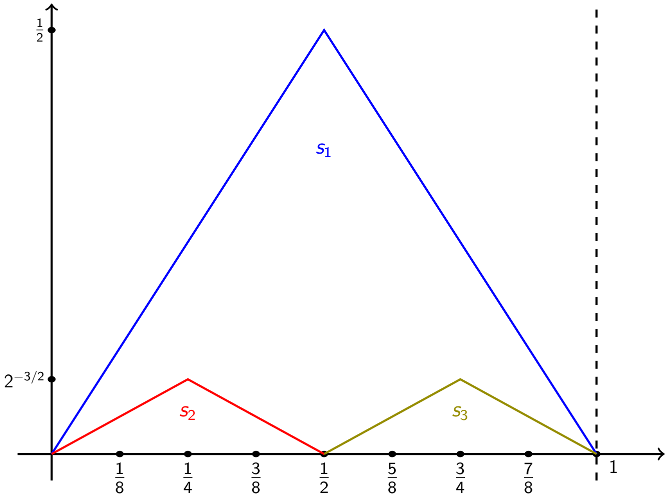
Functions $\{s_k : [0,1] \to \mathbb{R};\ k \geq 0\}$ *satisfy for* $2^n \leq k < 2^{n+1}$*:*

1. $Supp(s_k) = Supp(h_k) = [\frac{k-2^n}{2^n}, \frac{k-2^n+1}{2^n}]$
2. $\|s_k\|_\infty = \frac{1}{2^{n/2+1}}$

Schauder functions: illustration

$h_1 = \mathbb{1}_{[0,\frac{1}{2}]} - \mathbb{1}_{(\frac{1}{2},1]}$

$h_2 = \sqrt{2}\left(\mathbb{1}_{[0,\frac{1}{4}]} - \mathbb{1}_{(\frac{1}{4},\frac{1}{2}]}\right)$



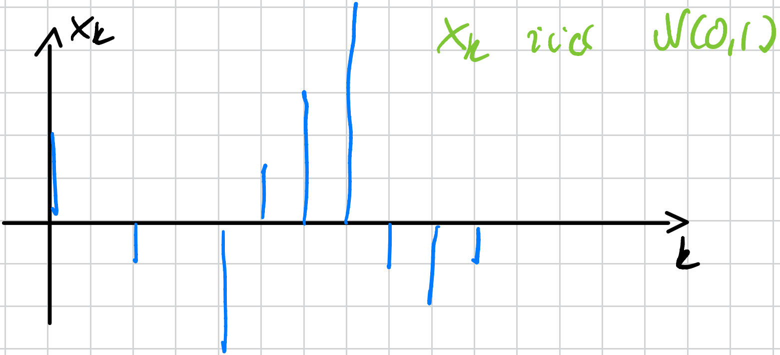
Spoiler: The space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we construct W is any space supporting a sequence

$$\boxed{\{ X_k ; \ k \geq 0 \}}$$

of i.i.d standard Gaussian r.v.

indep & identically distribution

This type of sequence might look like



Question: What is the order of magnitude of

$$\sup\{\, |X_k| \,;\, k \le n \}$$

Gaussian suprema

Lemma 10

Let $\{X_k;\, k \geq 1\}$ *i.i.d sequence of* $\mathcal{N}(0,1)$ *r.v. We set:*

$$M_n \equiv \sup\{|X_k|;\, 1 \leq k \leq n\}.$$

Then

$$M_n = O\left(\sqrt{\ln(n)}\right) \qquad \textit{almost surely}$$

slow growth!

$X_i \sim \mathcal{N}(0,1)$

Proof - Step 1 : Gaussian tails. Take $x > 0$

Then

$$\boxed{\mathbb{P}(\ |X_i| > x\ )}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{(-\infty,-x]\cup[x,\infty)} e^{-\frac{z^2}{2}}\, dz$$

$$= \frac{2}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{z^2}{2}}\, dz \qquad = e^{-\frac{z^2}{4}} \times e^{-\frac{z^2}{4}} \le e^{-\frac{x^2}{4}} e^{-\frac{z^2}{4}}$$

$$= c_1 e^{-\frac{x^2}{4}} \int_x^\infty e^{-\frac{z^2}{4}}\, dz$$

$$\le c_1 e^{-\frac{x^2}{4}} \int_0^\infty e^{-\frac{z^2}{4}}\, dz \qquad \boxed{\le c_2 e^{-\frac{x^2}{4}}}$$

Recall: To get information about the behavior of random sequences, we often use Borel-Cantelli.

Borel Cantelli: Let $\{A_k : k \geq 1\}$ be a sequence of events. Then

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$$

$$\Rightarrow \quad \boxed{\mathbb{P}(\underbrace{\limsup A_k}_{\bigcap_n \bigcup_{k \geq n} A_k \,=\, \text{"}A_k \text{ i.o"}}) = 0}$$

Application: Take

$$A_k = (|X_k| > a_k)$$

We have seen that

$$\mathbb{P}(A_k) \leq c\, e^{-\frac{a_k^2}{4}}$$

Choose a_k s.t. $\sum \mathbb{P}(A_k) < \infty$. For instance take a_k s.t.

$$\boxed{\mathbb{P}(A_k) < \frac{1}{k^4}}$$

<u>Recall</u> $\mathbb{P}(A_k) \leq c\, e^{-\frac{a_k^2}{4}}$

Take a_k s.t.

$$c\, e^{-\frac{a_k^2}{4}} \leq \frac{1}{k^4}$$

$$\Leftrightarrow \quad -\frac{a_k^2}{4} \leq -c_3 \ln(k)$$

$$\Leftrightarrow \quad \boxed{a_k \geq c_4 (\ln(k))^{\frac{1}{2}}}$$

We get: Borel-Cantelli applies

Conclusion: If

$$A_k = \left(|X_k| \geq c (\ln k)^{1/2} \right)$$

Then $\mathbb{P}(\limsup A_k) = 0$

$\Rightarrow$ a.s. we have $\tilde{c} = \tilde{c}(\omega)$

$$|X_k| \leq \tilde{c} (\ln k)^{1/2}$$

Or: $\exists\, n_0 = n_0(\omega)$ s.t. for all $n \geq n_0$,

$$\boxed{|X_k| \leq c (\ln k)^{1/2}}$$

Proof

Gaussian tail: Let $x > 0$. We have:

$$\begin{aligned} \mathbf{P}(|X_k| > x) &= \frac{2}{(2\pi)^{1/2}} \int_x^\infty e^{-\frac{z^2}{4}} e^{-\frac{z^2}{4}} \, dz \\ &\leq c_1 e^{-\frac{x^2}{4}} \int_x^\infty e^{-\frac{z^2}{4}} \, dz \leq c_2 e^{-\frac{x^2}{4}}. \end{aligned}$$

Application of Borel-Cantelli: Let $A_k = (|X_k| \geq 4(\ln(k))^{1/2})$.
According to previous step we have:

$$\mathbf{P}(A_k) \leq \frac{c}{k^4} \implies \sum_{k=1}^{\infty} \mathbf{P}(A_k) < \infty \implies \mathbf{P}(\limsup A_k) = 0$$

Conclusion: ω-a.s there exists $k_0 = k_0(\omega)$ such that
$\hookrightarrow |X_k(\omega)| \leq 4[\ln(k)]^{1/2}$ for $k \geq k_0$.