Concrete construction on $[0, 1]$

Proposition 11

Let

- $\{s_k;\ k \geq 0\}$ *Schauder functions family*
- $\{X_k;\ k \geq 0\}$ *i.i.d sequence of* $\mathcal{N}(0, 1)$ *random variables.*

We set:

$$W_t = \sum_{k \geq 0} X_k\, s_k(t).$$

Then W *is a Wiener process on* $[0, 1]$
$\hookrightarrow$ *In the sense of Definition 4.*

Step 1: Prove that

$$\sum_{k=0}^{\infty} X_k \, S_k(t)$$

is a convergent series in $C([0,1])$ (a.s.)

For this: set

$$\boxed{W_t^n = \sum_{k=0}^{2^n} X_k \, S_k(t)}$$

Is this a Cauchy sequence in $C([0,1])$?

We thus consider differences: for $m < n$,

$$\boxed{W_t^m - W_t^n = \sum_{k=2^m}^{2^n - 1} X_k \, S_k(t)}$$

Do we have (for arbitrary $\varepsilon > 0$)

$$\| W^m - W^n \|_\infty \leq \varepsilon$$

for $m > m_0(\omega)$?

Study of the finite sum : For $t \in [0,1]$,

$$\left| \sum_{k=2^m}^{2^n} X_k \, S_k(t) \right| \qquad \forall \eta > 0, \quad |X_k| \leq C_2 \, k^{\eta}$$

$$= \left| \sum_{p=m}^{n-1} \sum_{k=2^p}^{2^{p+1}-1} X_k \, \overbrace{S_k(t)}^{\geq 0} \right|$$

$$\leq \sum_{p=m}^{n-1} \sum_{k=2^p}^{2^{p+1}-1} \overbrace{|X_k|}^{\leq \; C_1 k^{\eta} \; \leq \; C_2 2^{p\eta}} \; S_k(t)$$

$$\leq C_3 \sum_{p=m}^{n-1} 2^{p\eta} \overbrace{\sum_{k=2^p}^{2^{p+1}} S_k(t)}^{\leq \frac{C_4}{2^{p/2}}}$$

Hyp: $\eta < \frac{1}{2}$

$$\leq C_5 \sum_{p=m}^{n-1} \frac{1}{2^{p(\frac{1}{2}-\eta)}} \leq \frac{C_6}{2^{m(\frac{1}{2}-\eta)}} \xrightarrow{m \to \infty} 0$$

Conclusion:

$$\sum_{k=0}^{\infty} X_k S_k \equiv W_t$$

is a convergent (a.s) series in $C([0,1])$

Next question: is the limit a Brownian motion

For free: (1) $W_0 = 0$

(4) W continuous

Step 2: Take $r < t$. Do we have

$$\delta W_{rt} \sim \mathcal{N}(0, t-r) \quad ?$$

Fact: Gaussian r.v. are good friends of characteristic functions. We have

$$\boxed{\begin{gathered} Z \sim \mathcal{N}(m, \sigma^2) \\ \Updownarrow \\ \mathbb{E}[e^{i\lambda Z}] = e^{i\lambda m} \, e^{-\frac{\sigma^2}{2}\lambda^2} \quad \forall \lambda \in \mathbb{R} \end{gathered}}$$

Reduced aim: prove $\forall \alpha \in \mathbb{R}$,

$$\boxed{\mathbb{E}\left[e^{i\alpha \, \delta W_{rt}} \right] = e^{-\alpha^2 \frac{(t-r)}{2}}}$$

We have

$$W_t = \sum_k X_k \, S_k(t)$$

$$\mathbb{E}\left[e^{i\alpha \, \delta W_{rt}} \right]$$

$$= \mathbb{E}\left[\exp\left(i\alpha \sum_k X_k \left(S_k(t) - S_k(r) \right) \right) \right]$$

$\perp\!\!\!\perp$

$$= \mathbb{E}\left\{ \prod_k \exp\left(i\alpha \, X_k \left(S_k(t) - S_k(r) \right) \right) \right\}$$

$$\overset{\perp\!\!\!\perp}{=} \prod_k \mathbb{E}\left[\exp\left(i\alpha \left(S_k(t) - S_k(r) \right) X_k \right) \right]$$

$$\mathbb{E}\left[e^{i\alpha\,\delta W_{rt}}\right]$$

$$= \prod_k \mathbb{E}\left[\exp\left(i\alpha\,(s_k(t)-s_k(r))\,\overbrace{X_k}^{\sim \mathcal{N}(0,1)}\right)\right]$$

$$= \prod_k \exp\left(-\frac{\alpha^2}{2}\,(s_k(t)-s_k(r))^2\right)$$

$$= \exp\left(-\frac{\alpha^2}{2}\sum_k (s_k(t)-s_k(r))^2\right)$$

$$= \exp\left(-\frac{\alpha^2}{2}\sum_k \left(s_k^2(t)-2\,s_k(t)s_k(r)+s_k(r)^2\right)\right)$$

Compute, for $r \le t$

Recall:
h_k onb of $L^2([0,1])$
$S_k = \int h_k$

$S_k(t) = \int_0^t h_k(s)\,ds = \int_0^1 h(s)\, \mathbb{1}_{[0,t]}(s)$

$$\boxed{\sum_{k=0}^{\infty} S_k(r)\, S_k(t)}$$

$$= \sum_{k=0}^{\infty} \langle h_k, \mathbb{1}_{[0,r]} \rangle \langle h_k, \mathbb{1}_{[0,t]} \rangle$$

$$= \langle \mathbb{1}_{[0,r]}, \mathbb{1}_{[0,t]} \rangle$$

$$= \int_0^1 \mathbb{1}_{[0,r]}(s)\, \mathbb{1}_{[0,t]}(s)\, ds \; \boxed{= r}$$

Conclusion $\quad \forall \alpha \in \mathbb{R}$

$$E[e^{i\alpha \delta W_{rt}}]$$

$$= \exp\left(-\frac{\alpha^2}{2} \sum_k \left(S_k^2(t) - 2 S_k(t) S_k(r) + S_k(r)^2\right)\right)$$

$$= \exp\left(-\frac{\alpha^2}{2} \left(t - 2r + r\right)\right)$$

$$= \exp\left(-\frac{\alpha^2}{2} (t-r)\right)$$

$$\Rightarrow \quad \boxed{\delta W_{rt} \sim \mathcal{N}(0, t-r)}$$

In the same way we can prove that if $r < u < t$,

$$\boxed{\delta W_{ru} \perp\!\!\!\perp \delta W_{ut}}$$

by proving

$$\mathbb{E}[\exp(i(\lambda_1 \delta W_{ru} + \lambda_2 \delta W_{ut}))]$$

$$= \mathbb{E}[\exp(i\lambda_1 \delta W_{ru})]\,\mathbb{E}[\exp(i\lambda_2 \delta W_{ut})]$$

$$\forall \lambda_1, \lambda_2 \in \mathbb{R}$$

Brightspace → Course Tools

→ Groups

Find your group

Email members of the group
(either 3 or 4 persons)

Thursday 9/10 : Field trip
to STEW 320
Nikos Zygouras (ICM speaker)
Talk at 3pm
Bonus +5 pts for attendance

Proof: uniform convergence

Step 1: Show that $\sum_{k\geq 0} X_k \, s_k(t)$ converges

$\hookrightarrow$ Uniformly in $[0,1]$, almost surely.

$\hookrightarrow$ This also implies that W is continuous a.s

Problem reduction: See that for all $\varepsilon > 0$

$\hookrightarrow$ there exists $n_0 = n_0(\omega)$ such that for all $n_0 \leq m < n$ we have:

$$\left\| \sum_{k=2^m}^{2^n-1} X_k \, s_k \right\|_\infty \leq \varepsilon.$$

Proof: uniform convergence (2)

Useful bounds:

1. Let $\eta > 0$. We have (Lemma 10):

$$|X_k| \leq c\, k^\eta, \quad \text{with} \quad c = c(\omega)$$

2. For $2^p \leq k < 2^{p+1}$, functions s_k have disjoint support. Thus

$$\left\| \sum_{k=2^p}^{2^{p+1}-1} \underbrace{s_k}_{\text{disjoint support}} \right\|_\infty \leq \frac{1}{2^{\frac{p}{2}+1}}.$$

Proof: uniform convergence (3)

Uniform convergence: for all $t \in [0,1]$ we have:

$$
\begin{aligned}
\left| \sum_{k=2^m}^{2^n} X_k \, s_k(t) \right| &\leq \sum_{p \geq m} \sum_{k=2^p}^{2^{p+1}-1} |X_k| \, s_k(t) \\
&\leq \sum_{p \geq m} \left(\sup_{2^p \leq \ell \leq 2^{p+1}-1} |X_\ell| \right) \left| \sum_{k=2^p}^{2^{p+1}-1} s_k(t) \right| \\
&\leq c_1 \sum_{p \geq m} \frac{1}{2^{p\left(\frac{1}{2}-\eta\right)}} \\
&\leq \frac{c_2}{2^{m\left(\frac{1}{2}-\eta\right)}},
\end{aligned}
$$

which shows uniform convergence.

Proof: law of δW_{rt}

Step 2: Show that $\delta W_{rt} \sim \mathcal{N}(0, t-s)$ for $0 \leq r < t$.

Problem reduction: See that for all $\lambda \in \mathbb{R}$,

$$\mathbf{E}\left[e^{\imath\lambda\,\delta W_{rt}}\right] = e^{-\frac{(t-r)\lambda^2}{2}}.$$

Recall: $\delta W_{rt} = \sum_{k\geq 0} X_k(s_k(t) - s_k(r))$

Computation of a characteristic function:
Invoking independence of X_k's and dominated convergence,

$$\begin{aligned}
\mathbf{E}\left[e^{\imath\lambda\,\delta W_{rt}}\right] &= \prod_{k\geq 0} \mathbf{E}\left[e^{\imath\lambda\,X_k(s_k(t)-s_k(r))}\right] \\
&= \prod_{k\geq 0} e^{-\frac{\lambda^2(s_k(t)-s_k(r))^2}{2}} = e^{-\frac{\lambda^2}{2}\sum_{k\geq 0}(s_k(t)-s_k(r))^2}
\end{aligned}$$

Proof: law of δW_{rt} (2)

Inner product computation: For $0 \le r < t$ we have

$$\sum_{k\geq 0} s_k(r)\, s_k(t) = \sum_{k\geq 0} \left\langle h_k,\, \mathbf{1}_{[0,r]} \right\rangle \left\langle h_k,\, \mathbf{1}_{[0,t]} \right\rangle = \left\langle \mathbf{1}_{[0,r]},\, \mathbf{1}_{[0,t]} \right\rangle = r.$$

Thus:

$$\sum_{k\geq 0} [s_k(t) - s_k(r)]^2 = t - r.$$

Computation of a characteristic function (2): We get

$$\mathbf{E}\left[e^{\imath\lambda\, \delta W_{rt}} \right] = e^{-\frac{\lambda^2}{2} \sum_{k\geq 0} (s_k(t) - s_k(r))^2} = e^{-\frac{(t-r)\lambda^2}{2}}.$$

Proof: increment independence

Simple case: For $0 \leq r < t$, we show that $W_r \perp\!\!\!\perp \delta W_{rt}$

Computation of a characteristic function: for $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\mathbf{E}\left[e^{\imath(\lambda_1 W_r + \lambda_2\, \delta W_{rt})}\right] = \prod_{k\geq 0} \mathbf{E}\left[e^{\imath\, X_k[\lambda_1 s_k(r) + \lambda_2(s_k(t) - s_k(r))]}\right]$$

$$= e^{-\frac{1}{2}\sum_{k\geq 0}[\lambda_1 s_k(r) + \lambda_2(s_k(t)-s_k(r))]^2} = e^{-\frac{1}{2}[\lambda_1^2 r + \lambda_2^2 (t-r)]}$$

Conclusion: We have, for all $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\mathbf{E}\left[e^{\imath(\lambda_1 W_r + \lambda_2\, \delta W_{rt})}\right] = \mathbf{E}\left[e^{\imath \lambda_1 W_r}\right]\, \mathbf{E}\left[e^{\imath \lambda_2\, \delta W_{rt}}\right],$$

and thus $W_r \perp\!\!\!\perp \delta W_{rt}$.

Effective construction on $[0, \infty)$

Proposition 12

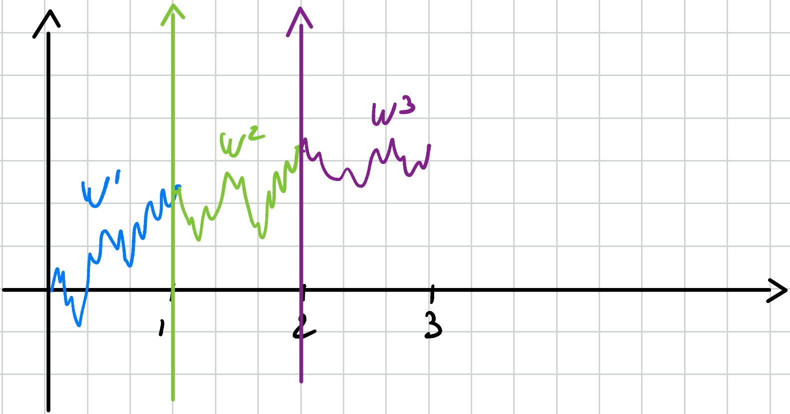
Let:

- *For* $k \geq 1$, *a space* $(\Omega_k, \mathcal{F}_k, \mathbf{P}_k)$
  $\hookrightarrow$ *On which a Wiener process* W^k *on* $[0, 1]$ *is defined*
- $\bar{\Omega} = \prod_{k \geq 1} \Omega_k$, $\bar{\mathcal{F}} = \otimes_{k \geq 1} \mathcal{F}_k$, $\bar{\mathbf{P}} = \otimes_{k \geq 1} \mathbf{P}_k$

We set $W_0 = 0$ *and recursively:*

$$W_t = W_n + W_{t-n}^{n+1}, \quad \textit{if} \quad t \in [n, n+1].$$

Then W *is a Wiener process on* $\mathbb{R}_+$
$\hookrightarrow$ *Defined on* $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}})$.



Used: If $X_1 \sim \mathcal{N}(0, \sigma_1^2)$ $X_1 \perp\!\!\!\perp X_2$
$X_2 \sim \mathcal{N}(0, \sigma_2^2)$

$$\Rightarrow \boxed{X_1 + X_2 \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)}$$

Partial proof

Aim: See that $\delta W_{st} \sim \mathcal{N}(0, t-s)$
$\hookrightarrow$ with $m \le s < m+1 \le n \le t < n+1$

Decomposition of δW_{st}: We have

$$\delta W_{st} = \sum_{k=1}^{n} W_1^k + W_{t-n}^{n+1} - \left(\sum_{k=1}^{m} W_1^k + W_{s-m}^{m+1} \right) = Z_1 + Z_2 + Z_3,$$

with

$$Z_1 = \sum_{k=m+2}^{n} W_1^k, \quad Z_2 = W_1^{m+1} - W_{s-m}^{m+1}, \quad Z_3 = W_{t-n}^{n+1}.$$

Law of δW_{st}: The Z_j's are independent centered Gaussian.
Thus $\delta W_{st} \sim \mathcal{N}(0, \sigma^2)$, with:

$$\sigma^2 = n - (m+1) + 1 - (s-m) + t - n = t - s.$$

Wiener process in $\mathbb{R}^n$

Definition 13

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space
- $\{W_t;\ t \geq 0\}$ $\mathbb{R}^n$-valued stochastic process

We say that W is a Wiener process if:

1. $W_0 = 0$ almost surely
2. Let $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n$. The increments
$$\delta W_{t_0 t_1}, \delta W_{t_1 t_2}, \ldots, \delta W_{t_{n-1} t_n} \quad \text{are independent}$$
3. For $0 \leq s < t$ we have $\delta W_{st} \sim \mathcal{N}(0, (t-s)\mathrm{Id}_{\mathbb{R}^n})$
4. W continuous paths, almost surely

Remark: One can construct W

$\hookrightarrow$ from n independent real valued Brownian motions.

Rmk Consider $W^1, \ldots, W^n$ n indep. Brownian motions

Define $W_t = \begin{pmatrix} W_t^1 \\ W_t^2 \\ \vdots \\ W_t^n \end{pmatrix} \in \mathbb{R}^n$

Then W is a $\mathbb{R}^n$-valued Wiener process

Illustration: 2-d Brownian motion

Durrett