Wiener process in a filtration

Definition 14

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ stochastic basis
- $\{W_t;\ t \geq 0\}$ stochastic process with values in $\mathbb{R}^n$

We say that W is a Wiener process
with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ if:

1. $W_0 = 0$ almost surely, and $\boxed{W_t \in \mathcal{F}_t}$
2. Let $0 \leq s < t$. Then $\boxed{\delta W_{st} \perp\!\!\!\perp \mathcal{F}_s.}$
3. For $0 \leq s < t$ we have $\delta W_{st} \sim \mathcal{N}(0, (t-s)\mathrm{Id}_{\mathbb{R}^n})$
4. W has continuous paths almost surely

Remark: A Wiener process according to Definition 13 is a Wiener process in its natural filtration.

Rmk: If X r.v $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$

$\mathcal{G}$ σ-algebra

we say that $X \perp\!\!\!\perp \mathcal{G}$ if

$$\sigma(X) \perp\!\!\!\perp \mathcal{G}$$

Otherwise stated: For $\begin{cases} G \in \mathcal{G} \\ B \in \mathcal{B} \end{cases}$,

$$\boxed{\mathbb{P}((X \in B) \cap G) = \mathbb{P}(X \in B)\,\mathbb{P}(G)}$$

Brown

Some facts about Brown:

- Scottish, Lived 1773-1858
- Pioneer in the use of microscope
- Detailed description of cell nucleus
- Observed 2-d Brownian motion
  $\hookrightarrow$ Pollen particles in water

Outline

3 desirable properties for stoch process

Gaussian

Martingale

Markov

$\boxed{\text{BM is all 3 !}}$

Gaussian processes

Definition 15

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space
- $\{X_t;\ t \geq 0\}$ stochastic process, with values in $\mathbb{R}$

We say that X is a Gaussian process if
for all $0 \leq t_1 < \cdots < t_n$ we have:

$$(X_{t_1}, \ldots, X_{t_n}) \quad \text{Gaussian vector.}$$

Rmk Let $Y = (Y^1, \dots, Y^n)$ be a random vector. We say that Y is a Gaussian vector if

$Y^1, Y^2, \dots, Y^n$ are all Gaussian

AND $\forall\, \lambda_1, \dots, \lambda_n \in \mathbb{R}$ we have

$\sum_{i=1}^{n} \lambda_i Y^i$ is Gaussian

Facts about Gauss. vectors

(1) It is easy to construct

$Y = (Y^1, Y^2)$

with $Y^1 \sim \mathcal{N}(0,1)$, $Y^2 \sim \mathcal{N}(0,1)$
but

$\boxed{Y \text{ not Gauss. vector}}$

Take

$\mathbb{P}(\varepsilon = \pm 1) = \frac{1}{2}$

$Y^1 \sim \mathcal{N}(0,1)$ $\qquad$ $\varepsilon \sim$ Rademacher

$\varepsilon \perp\!\!\!\perp Y^1$. Then $Y^2 = \varepsilon Y^1$

check that $Y = (Y^1, Y^2)$ not Gauss. vect

(2) Basic example of Gaussian vector:

$$Y = \begin{pmatrix} Y^1 \\ \vdots \\ Y^n \end{pmatrix} \quad \text{with } Y^i \text{ iid } \mathcal{N}(0,1)$$

Then any Gaussian vector in $\mathbb{R}^n$ can be written (in distribution) as

$$Z = MY + b$$

with $M \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$

↳ Gaussian vectors are good friends of linear algebra

(3) A Gauss. vector Y is characterized (in distribution) by

$$\boxed{\begin{aligned} m &= \mathbb{E}[Y] \\ \Gamma &= \mathbb{E}[(Y-m)(Y-m)^t] \end{aligned}}$$

or $\Gamma^{ij} = \mathrm{cov}(Y^i, Y^j)$

$\hookrightarrow$ $Y \sim \mathcal{N}(m, \Gamma)$

(4) Characteristic function of

$Y \sim \mathcal{N}(m, \Gamma)$

is given by $(\lambda \in \mathbb{R}^n)$

$$\boxed{\begin{aligned} &\mathbb{E}\left[e^{i\langle\lambda, Y\rangle}\right] \\ &= e^{i\langle\lambda, m\rangle} \exp\left(-\frac{1}{2}\lambda^t \Gamma \lambda\right) \end{aligned}}$$

Gaussian property

$= \min(s,t)$

Theorem 16

Let W be a real valued Brownian motion.
Then W is a Gaussian process, with covariance

$$\mathbf{E}[W_s W_t] = s \wedge t, \quad \text{for all } s, t \geq 0.$$

$\hookrightarrow$ Brownian motion in the sense of Definition 4.
$\hookrightarrow$ Gaussian process in the sense of Definition 15.

Step 1 We have to prove, for

$0 \leq t_1 < t_2 \cdots < t_n$

that

$Y = (W_{t_1}, \dots, W_{t_n})$ is a Gauss. vector

Note

We already know that $Y^1, \dots, Y^n$ are all Gaussian r.v.

Easy to check: Gauss + $\perp\!\!\!\perp$

$$z = (\delta W_{0 t_1}, \delta W_{t_1 t_2}, \dots, \delta W_{t_{n-1} t_n})$$

is a Gauss. vector

Then one can write

$$Y = M z$$

Ex: find proper M

for a matrix $M \in \mathbb{R}^{n \times n}$.

Thus Y is also a Gauss. vector.

Ex: $W_{t_2} = W_{t_1} + \delta W_{t_1 t_2}$

Conclusion of Step 1: W is a Gauss. process.

$W_s \sim \mathcal{N}(0, s)$

Step 2. Mean & covariance function.

Mean: $\mathbb{E}[W_t] = 0 \equiv \mu_t$

Cov function: for $s \le t$

$$\rho_{s,t} \equiv \mathrm{Cov}(W_s, W_t) = \mathbb{E}[W_s W_t]$$

$$= \mathbb{E}[W_s (W_s + \delta W_{st})]$$

$$= \mathbb{E}[W_s^2] + \mathbb{E}[W_s \, \delta W_{st}] = s + 0$$

We have obtained: If $s \leq t$

$$\rho_{st} = s = s \wedge t$$

Thus

$$\boxed{\rho_{st} = s \wedge t}$$

Proof

Notation: For $0 = t_0 \leq t_1 < \cdots < t_n$ we set

- $X_n = (W_{t_1}, \ldots, W_{t_n})$
- $Y_n = (\delta W_{t_0 t_1}, \ldots, \delta W_{t_{n-1} t_n})$

Vector Y_n: Thanks to independence of increments of W
$\hookrightarrow$ Y_n is a Gaussian vector.

Vector X_n: There exists $M \in \mathbb{R}^{n,n}$ such that $X_n = M Y_n$
$\hookrightarrow$ X_n Gaussian vector

Covariance matrix: We have $\mathbf{E}[W_s W_t] = s \wedge t$. Thus

$$(W_{t_1}, \ldots, W_{t_n}) \sim \mathcal{N}(0, \Gamma_n), \quad \text{with} \quad \Gamma_n^{ij} = t_i \wedge t_j.$$

Consequence of Gaussian property

Characterization of a Gaussian process:

Let X Gaussian process. The law of X is characterized by:

$$\mu_t = \mathbf{E}[X_t], \quad \text{and} \quad \rho_{s,t} = \mathbf{Cov}(X_s, X_t).$$

Another characterization of Brownian motion:

Let W real-valued Gaussian process with

$$\mu_t = 0, \quad \text{and} \quad \rho_{s,t} = s \wedge t.$$

Then W is a Brownian motion.

Brownian scaling

Proposition 17

Let

- W *real-valued Brownian motion.*
- *A constant* $a > 0$.

We define a process W^a *by:*

$$W_t^a = a^{-1/2} W_{at}, \quad \textit{for} \quad t \geq 0.$$

Then W^a *is a Brownian motion.*

Proof:

Gaussian characterization of Brownian motion.

$W^a_t = a^{-1/2} W_{at}$

Consider the process W^a. Then

(i) W^a is a Gaussian process:

$(W^a_{t_1}, \ldots, W^a_{t_n})$ Gauss. vector check!

(ii) $\boxed{\mu_t} \equiv \mathbb{E}[W^a_t]$

$= \mathbb{E}[a^{-1/2} W_{at}]$

$= a^{-1/2}\, \mathbb{E}[W_{at}]$

$\boxed{= 0}$

(iii)

$s \leq t$

$a > 0$

$$\boxed{\rho_{st}} = \mathbb{E}[a^{-\frac{1}{2}} W_{as} \cdot a^{-\frac{1}{2}} W_{at}]$$

$$= \frac{1}{a} \mathbb{E}[W_{as} W_{at}]$$

$$= \frac{1}{a} \min(as, at)$$

$$= \frac{as}{a}$$

$$= s \quad \boxed{= s \wedge t}$$

$\Rightarrow \{W_t^a \ ; \ t \geq 0\}$ is a Wiener process

Canonical space

Aim: define the $\mathcal{L}(W)$ as a prob. measure on a functional space

Lemma 18

Let $E = \mathcal{C}([0,\infty);\mathbb{R}^n)$. *We set:*

$\|f\|_\infty = \infty$ for $f = W$

$$d(f,g) = \sum_{k\geq 1} \frac{\|f-g\|_{\infty,k}}{2^k\left(1+\|f-g\|_{\infty,k}\right)}$$

where

$$\|f-g\|_{\infty,k} = sup\left\{|f_t - g_t|;\ t\in[0,k]\right\}.$$

Then E *is a separable complete metric space.*

Rmk: This distance quantifies "convergence on compact intervals)

Question : Can we consider

$X = \{X_t ; t \in T\}$ as

$X \in \mathbb{R}^T$?

Then question 2 :

(1) σ - algebra ?

(2) Is $\|X\|_\infty$ a r.v ?

(3) Convergence of stoch processes ?