

Canonical space

Quantifies convergence of $f_n \rightarrow f$
s.t. $\|f_n - f\|_{\infty, [0, k]} \rightarrow 0$

Lemma 18

Let $E = \mathcal{C}([0, \infty); \mathbb{R}^n)$. We set:

$$d(f, g) = \sum_{k \geq 1} \frac{\|f - g\|_{\infty, k}}{2^k (1 + \|f - g\|_{\infty, k})}$$

where

$$\|f - g\|_{\infty, k} = \sup \{|f_t - g_t|; t \in [0, k]\}.$$

Then E is a separable complete metric space.

Question: we might want to consider (W random function)

$$W : (\Omega, \mathcal{F}) \longrightarrow (E, \mathcal{B}) \xrightarrow{\text{Borel}}$$

as a r.v. However, what we know is measurability of

$$\{\omega; W_{t_1}(\omega) \in A_1, \dots, W_{t_k}(\omega) \in A_k\}$$

$$\text{for } A_1, \dots, A_k \in \mathcal{B}(\mathbb{R}^n) \\ t_1 < t_2 < \dots < t_k$$

Question: is this compatible?

Borel σ -algebra on E

Proposition 19

Let $E = \mathcal{C}([0, \infty); \mathbb{R}^n)$. For $m \geq 1$ we consider:

- $0 \leq t_1 < \dots < t_m$
- $A_1, \dots, A_m \in \mathcal{B}(\mathbb{R}^n)$

Let $\mathcal{A}$ be the σ -algebra generated by rectangles:

$$R_{t_1, \dots, t_m}(A_1, \dots, A_m) = \{x \in E; x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\}.$$

Then $\mathcal{A} = \mathcal{B}(E)$, Borel σ -algebra on E .

Wiener measure

Theorem 20

Let

- W $\mathbb{R}^n$ -valued Wiener process, defined on $(\Omega, \mathcal{F}, \mathbf{P})$.
- $T : (\Omega, \mathcal{F}) \rightarrow (E, \mathcal{A})$, such that $T(\omega) = \{W_t(\omega); t \geq 0\}$.

Then:

- 1 The application T is measurable
- 2 Let $\mathbf{P}_0 = \mathbf{P} \circ T^{-1}$, measure on $(E, \mathcal{A})$.
 $\mathbf{P}_0$ is called **Wiener measure**.
- 3 Under $\mathbf{P}_0$, the canonical process ω can be written as:

$$\omega_t = W_t, \quad \text{where } W \text{ Brownian motion.}$$

Proof

Inverse image of rectangles: We have

$$T^{-1}(R_{t_1, \dots, t_m}(A_1, \dots, A_m)) = (W_{t_1} \in A_1, \dots, W_{t_m} \in A_m) \in \mathcal{F}.$$

Conclusion: T measurable, since $\mathcal{A}$ generated by rectangles.

Einstein

Einstein and Brownian motion:

- In 1905, 4 revolutionary papers
 - ↔ While Einstein was employee at Patent Office in Zurich
- One of the 4 papers explains BM
 - ↔ Action of water molecules on pollen
- Lead to evidence for atoms
- First elements about transition kernel of Brownian motion



Outline

- 1 Stochastic processes
- 2 Definition and construction of the Wiener process
- 3 First properties
- 4 Martingale property**
- 5 Markov property
- 6 Pathwise properties

Martingale prop. X is a martingale
if

(i) $X_t \in L^1(\Omega)$

(ii) $X_t \in \mathcal{F}_t$

(iii) $E[X_t | \mathcal{F}_s] = X_s$

Best guess for X_s in the future after
s is constant ($= X_s$)

Martingales

Definition 21

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ stochastic basis
- $\{X_t; t \geq 0\}$ stochastic process, with values in $\mathbb{R}^n$

We say that X is a $\mathcal{F}_t$ -martingale if

- 1 $X_t \in L^1(\Omega)$ for all $t \geq 0$.
- 2 X is $\mathcal{F}_t$ -adapted.
- 3 $\mathbf{E}[\delta X_{st} | \mathcal{F}_s] = 0$ for all $0 \leq s < t$.

$$X_t - X_s$$

Martingale property

Proposition 22

*Let W a $\mathcal{F}_t$ -Brownian motion.
Then W is a $\mathcal{F}_t$ -martingale.*

$\hookrightarrow$ Brownian motion in the sense of Definition 14.

$\hookrightarrow$ Martingale in the sense of Definition 21.

Remark: If W is real valued, then $M_t = W_t^2 - t$ is also a $\mathcal{F}_t$ -martingale.

Proof that W is a $\mathbb{F}_t$ -MG

(i) $W_t \in L^1(\Omega)$ for all $t \geq 0$

$$\mathbb{E}[|W_t|] < \infty$$

Possible strategy

$$\int_0^\infty \mathbb{P}(|W_t| \geq x) dx = \mathbb{E}[|W_t|]$$

Other argument $W_t \sim \mathcal{N}(0, t)$ ($t > 0$)

$$\begin{aligned} \text{Thus } \mathbb{E}[|W_t|] &= \int_{\mathbb{R}^n} |x| \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{n/2}} dx \\ &< \infty \quad (\text{conv. integral}) \end{aligned}$$

$$(ii) \quad W_t \in \mathcal{F}_t$$

(part of the def of $\mathcal{F}_t$ - BM)

$$(iii) \quad \mathbb{E} [\overbrace{\Delta W_{st}}^{\perp \mathcal{F}_s} \mid \mathcal{F}_s]$$

$$= \mathbb{E} [\Delta W_{st}]$$

$$= 0 \quad (W_0, W_s \text{ centered})$$

Thus W is a martingale

Ex: Prove that

$$M_t \equiv W_t^2 - t$$

is a martingale

Let $T: (\Omega, \mathcal{F}, (\mathcal{F}_t)) \xrightarrow{r.v.} [0, \infty]$

Then T stopping time if

$$(T \leq t) \in \mathcal{F}_t$$

Interpretation:

$T \equiv$ time when an investor decides to sell

↳ If you decide to sell before t ,
it is based on your information before t

Proof

We check the three items of Definition 21.

Integrability: Items (1) and (3) in Definition 14 give $W_t = \delta W_{0t} \sim \mathcal{N}(0, t \text{Id}_{\mathbb{R}^n})$. Therefore

$$\mathbf{E} [|W_t|] < \infty, \quad \text{that is} \quad W_t \in L^1(\Omega).$$

Adaptation: The process W is $\mathcal{F}_t$ -adapted
 $\hookrightarrow$ Assumed in Definition 14.

Increments: Let $0 \leq s < t$. Item (2) in Definition 14 states that $\delta W_{st} \perp\!\!\!\perp \mathcal{F}_s$. Hence

$$\mathbf{E} [\delta W_{st} \mid \mathcal{F}_s] = \mathbf{E} [\delta W_{st}] = 0,$$

where the last identity stems from $\delta W_{st} \sim \mathcal{N}(0, (t-s)\text{Id}_{\mathbb{R}^n})$.

Stopping time

Definition 23

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ stochastic basis.
- S random variable, with values in $[0, \infty]$.

We say that S is a stopping time if for all $t \geq 0$ we have:

$$(S \leq t) \in \mathcal{F}_t$$

Interpretation 1: If we know $X_{[0,t]}$

$\hookrightarrow$ One also knows if $S \leq t$ or $S > t$

Interpretation 2: $S \equiv$ instant for which one stops playing

$\hookrightarrow$ Only depends on information up to current time.

Typical examples of stopping time

Theorem 24

Let:

- X process with values in $\mathbb{R}^d$, $\mathcal{F}_t$ -adapted and continuous.
- G open set in $\mathbb{R}^d$.
- F closed set in $\mathbb{R}^d$.

We set:

$$T_G = \inf \{t \geq 0; X_t \in G\}, \quad T_F = \inf \{t \geq 0; X_t \in F\}.$$

Then:

- T_F is a stopping time.
- T_G is a stopping time when X is a Brownian motion.

Proof for T_G

First aim: prove that for $t > 0$ we have

$$(T_G < t) \in \mathcal{F}_t \quad (1)$$

Problem reduction for (1): We show that

$$(T_G < t) = \bigcup_{s \in \mathbb{Q} \cap [0, t)} (X_s \in G). \quad (2)$$

Since $\bigcup_{s \in \mathbb{Q} \cap [0, t)} (X_s \in G) \in \mathcal{F}_t$, this proves our claim.

First inclusion for (2):

$$\bigcup_{s \in \mathbb{Q} \cap [0, t)} (X_s \in G) \subset (T_G < t): \text{ trivial.}$$

Proof for T_G (2)

Second inclusion for (2): If $T_G < t$, then

- There exist $s < t$ such that $X_s \in G$. We set $X_s \equiv x$.
- Let $\varepsilon > 0$ such that $B(x, \varepsilon) \in G$

Then:

- There exists $\delta > 0$ such that $X_r \in B(x, \varepsilon)$ for all $r \in (s - \delta, s + \delta)$.
- In particular, there exists $q \in \mathbb{Q} \cap (s - \delta, s]$ such that $X_q \in G$.

Since

$$(\mathbb{Q} \cap (s - \delta, s]) \subset (\mathbb{Q} \cap [0, t]),$$

we have the second inclusion.

Proof for T_G (3)

Optional times: We say that $T : \Omega \rightarrow [0, \infty]$ is an optional time if

$$(T < t) \in \mathcal{F}_t.$$

Remark: Relation (1) proves that T_G is optional.

Optional times and stopping times:

- A stopping time is an optional time.
- An optional time satisfies $(T \leq t) \in \mathcal{F}_{t+} \equiv \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$.
- When X is a Brownian motion, $\mathcal{F}_{t+} = \mathcal{F}_t$ (by Markov prop.).
- When X is a Brownian motion, **optional time = stopping time**.

Conclusion: When X is a Brownian motion, T_G is a stopping time.

Sup-Inf of stopping times

Proposition 25

Let S and T two stopping times. Then

① $S \wedge T$

② $S \vee T$

are stopping times.

Applications: Z constant $\Rightarrow T \wedge Z$
 T stopping time $\Rightarrow$ stopping time

Constant stopping times

Proposition 26

Let T be a deterministic time, that is there exists $t_0 \geq 0$ such that

$$T = t_0, \text{ almost surely.}$$

Then T is a stopping time.

Proof $(T \leq t)$ is either Ω or $\emptyset$
This is always $\mathcal{F}_t$ -meas.

Information at time S

Definition 27

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ stochastic basis.
- S stopping time.

The σ -algebra $\mathcal{F}_S$ is defined by:

$$\mathcal{F}_S = \{A \in \mathcal{F}; [A \cap (S \leq t)] \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Interpretation:

$\mathcal{F}_S \equiv$ Information up to time S .

Ann of optional sampling

If X is a martingale, then

$$\mathbb{E}[X_t | \mathcal{F}_\Delta] = X_\Delta$$

Q: Can we extend this to random times

A: Yes, if $t \rightarrow T$ stopping time and (X_t) slightly more than L^1

Optional sampling theorem

Theorem 28

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ stochastic basis.
- S, T two stopping times, with $S \leq T$. (a.s.)
- X continuous martingale.

Hypothesis:

- $\{X_{t \wedge T}; t \geq 0\}$ **uniformly integrable** martingale.

Then:

$$\mathbf{E}[X_T | \mathcal{F}_S] = X_S.$$

In particular:

$$\mathbf{E}[X_T] = \mathbf{E}[X_S] = \mathbf{E}[X_0].$$

$(X_t; t \geq 0)$ uniformly integrable:

, $\forall \varepsilon$, $\exists \kappa$ s.t.

$$\sup_{t \geq 0} \mathbb{E}[|X_t| \mathbb{1}_{(|X_t| \geq \kappa)}] \leq \varepsilon$$

Main example of UI: when

$$\sup_{t \geq 0} \mathbb{E}[|X_t|^p] < \infty$$

with $p > 1$

Remarks

Strategy of proof:

- One starts from known discrete time result.
- X is approximated by a discrete time martingale

$$Y_m \equiv X_{t_m}, \quad \text{with} \quad t_m = \frac{m}{2^n}.$$

Checking the assumption: Set $Y_t = X_{t \wedge T}$.

$\{Y_t; t \geq 0\}$ uniformly integrable martingale in following cases:

- $|Y_t| \leq M$ with M deterministic constant independent of t .
- $\sup_{t \geq 0} \mathbf{E}[|Y_t|^2] \leq M$.
- $\sup_{t \geq 0} \mathbf{E}[|Y_t|^p] \leq M$ with $p > 1$.

Example of stopping time computation

Proposition 29

Let:

- B standard Brownian motion, with $B_0 = 0$.
- $-a < 0 < b$
- $T_a = \inf\{t \geq 0 : B_t = -a\}$ and $T_b = \inf\{t \geq 0 : B_t = b\}$.
- $T = T_a \wedge T_b$.

Then:

$$\mathbf{P}(T_a < T_b) = \frac{b}{b+a}, \quad \text{and} \quad \mathbf{E}[T] = ab.$$

