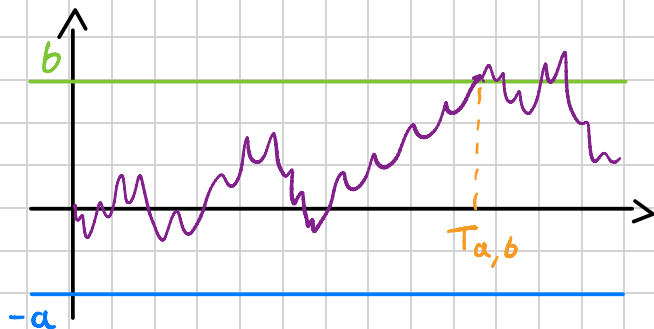


Optional sampling thm: For T stopping time

$$\mathbb{E}[M_T] = \mathbb{E}[M_0] \quad \text{under some integrability conditions}$$



closed set: $\{a, b\}$

Example of stopping time computation

Proposition 29

Let:

- B standard Brownian motion, with $B_0 = 0$.
- $-a < 0 < b$
- $T_a = \inf\{t \geq 0 : B_t = -a\}$ and $T_b = \inf\{t \geq 0 : B_t = b\}$.
- $T = T_a \wedge T_b$.

Then:

$$\mathbf{P}(T_a < T_b) = \frac{b}{b+a}, \quad \text{and} \quad \mathbf{E}[T] = ab.$$

We apply optional sampling to

$$M = W$$

$$T = T_a \wedge T_b$$

Then we wish to exploit

$$\mathbb{E}[W_T] \stackrel{?}{=} \mathbb{E}[W_0] = 0$$

This is true (optional sampling) if

$$\{N_t \equiv W_{t \wedge T}; t \geq 0\} \quad \cup \mathcal{I}$$

Recall

$\{N_t; t \geq 0\}$ UI whenever

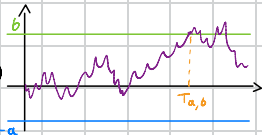
$$(*) \sup_{t \geq 0} E[|N_t|^p] < \infty, \text{ with } p > 1$$

Here we have $N_t = W_{t \wedge T}$

$$\Rightarrow |N_t| \leq \max\{a, b\}$$

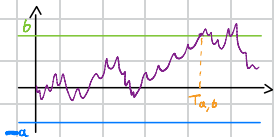
$$\Rightarrow (*) \text{ satisfied for } p = \infty$$

$$\Rightarrow (*) \text{ satisfied for all } p > 1$$



Admitted: $T < \infty$ a.s.

Optional sampling:



$$0 = \mathbb{E}[W_T]$$

$$= \mathbb{E}[-a \mathbb{1}_{(T_a < T_b)} + b \mathbb{1}_{(T_a > T_b)}]$$

$$= -a \mathbb{P}(T_a < T_b) + b \mathbb{P}(T_a > T_b)$$

$\Rightarrow \mathbb{P}(T_a < T_b), \mathbb{P}(T_a > T_b)$ solve the system

$$\begin{cases} -a x + b y = 0 \\ x + y = 1 \end{cases} \Rightarrow$$

$$\mathbb{P}(T_a < T_b) = \frac{b}{a+b}$$

For $E(T)$: Apply optional sampling to

$$M_t = W_t^2 - t \quad W_{t \wedge T}^2$$

Question: Do we have

$$\{M_{t \wedge T} ; t \geq 0\} \quad \text{UI?}$$

Recall: $M_T = W_T^2 - T$

Pb: $T < \infty$ a.s., but we might have
 $E[T] = \infty$ (ex: Cauchy distribution)

Approximation : Define $T^{(n)} = T \wedge n$
for $n \in \mathbb{N}$

Then $T^{(n)}$ is a bounded
stopping time. Therefore

$$\mathbb{E}[M_{T^{(n)}}] = 0$$

$$\Rightarrow \mathbb{E}[W_{T^{(n)}}^2 - T^{(n)}] = 0$$

$$\Rightarrow \mathbb{E}[W_{T \wedge n}^2] = \mathbb{E}[T \wedge n]$$

Taking limits

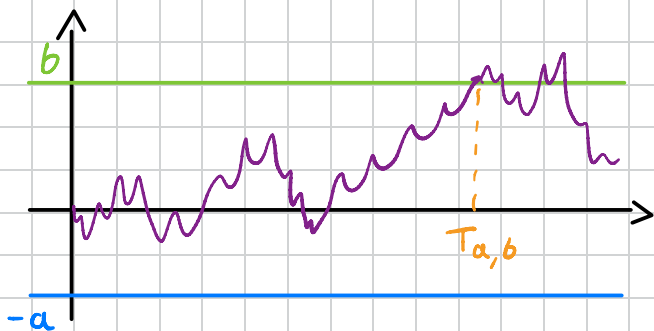
$$\begin{array}{ccc} \mathbb{E}[W_{T \wedge n}^2] & = & \mathbb{E}[T \wedge n] \\ \downarrow^{n \rightarrow \infty} & & \downarrow^{n \rightarrow \infty} \\ \text{dom} & & \text{monotone} \\ \text{convergence} & & \text{convergence} \\ \mathbb{E}[W_T^2] & = & \mathbb{E}[T] \end{array}$$

LHS: $\mathbb{E}[W_T^2]$

$$= \mathbb{E}[a^2 \mathbb{1}_{T_a < T_b} + b^2 \mathbb{1}_{(T_a > T_b)}]$$

$$= a^2 \mathbb{P}(T_a < T_b) + b^2 \mathbb{P}(T_a > T_b)$$

$$\vdots$$
$$= ab = \mathbb{E}[T] \text{ (RHS)}$$



Rmk By monotone convergence, we have

$$\boxed{E[T_b]} = \lim_{a \rightarrow \infty} \overbrace{E[T_{a,b}]}^{ab} \boxed{= \infty}$$

Proof

Optional sampling for $M_t = B_t$: yields

$$\mathbf{P}(T_a < T_b) = \frac{b}{b+a}.$$

Optional sampling for $M_t = B_t^2 - t$: yields, for a constant $\tau > 0$,

$$\mathbf{E}[B_{T \wedge \tau}^2] = \mathbf{E}[T \wedge \tau].$$

Limiting procedure: by dominated and monotone convergence,

$$\mathbf{E}[B_T^2] = \mathbf{E}[T].$$

Conclusion: we get

$$\mathbf{E}[T] = \mathbf{E}[B_T^2] = a^2 \mathbf{P}(T_a < T_b) + b^2 \mathbf{P}(T_b < T_a) = ab.$$

Bachelier

Bachelier and Brownian motion:

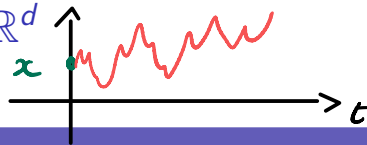
- Born 1870, son of a wine merchant
 $\hookrightarrow$ Used to business
- In 1900 (predates Einstein) PhD thesis
 $\hookrightarrow$ Theory of speculation
- Mathematical description of BM
 $\hookrightarrow$ Application to stock pricing
- Not well perceived by academic establishment



Outline

- 1 Stochastic processes
- 2 Definition and construction of the Wiener process
- 3 First properties
- 4 Martingale property
- 5 Markov property**
- 6 Pathwise properties

Wiener measure indexed by $\mathbb{R}^d$



Proposition 30

Let:

- $x \in \mathbb{R}^d$.

There exists a probability measure $\mathbf{P}_x$ on $(E, \mathcal{A})$ such that:

$\hookrightarrow$ Under $\mathbf{P}_x$ the canonical process ω can be written as:

$$\omega_t = x + W_t, \quad \text{where } W \text{ Brownian motion.}$$

Notations:

- We consider $\{\mathbf{P}_x; x \in \mathbb{R}^d\}$.
- Expected value under $\mathbf{P}_x$: $\mathbf{E}_x$.

Shift on paths

Definition 31

Let

- $E = \mathcal{C}([0, \infty); \mathbb{R}^d)$, equipped with Borel $\mathcal{A} \equiv \sigma$ -algebra.
- $t \geq 0$.

We set:

$$\theta_t : E \rightarrow E, \quad \{\omega_s; s \geq 0\} \mapsto \{\omega_{t+s}; s \geq 0\}$$

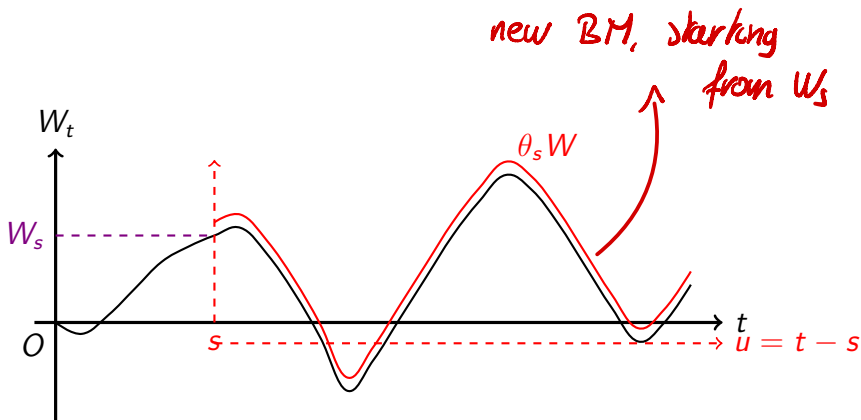
Shift and future: Let $Y : E \rightarrow \mathbb{R}$ measurable.

$\hookrightarrow$ Then $Y \circ \theta_s$ depends on future after s .

Example: For $n \geq 1$, f measurable and $0 \leq t_1 < \dots < t_n$,

$$Y(\omega) = f(\omega_{t_1}, \dots, \omega_{t_n}) \quad \implies \quad Y \circ \theta_s = f(W_{s+t_1}, \dots, W_{s+t_n}).$$

Shift on paths: illustration



Red curve: $u \mapsto (\theta_s W)_u = W_{s+u}$, clock re-based at s

We want to shift W in time
 $(\Theta_s W)$. We also want to
shift functions of W .

Notation: Take $t_1 \geq 0$. Consider

$$Y = f(W_{t_1})$$

Then

$$\Theta_s Y = f(W_{s+t_1})$$

In the same way

$$Y = f(W_{t_1}, \dots, W_{t_n})$$

$$\Rightarrow \Theta_s Y = f(W_{s+t_1}, \dots, W_{s+t_n})$$

For more general r.v.:

$$\Theta_s Y \equiv Y \circ \Theta_s$$

Example: $Y = \sup_{t \in [2,5]} W_t$

Markov property

Interpretation: After time s ,
 W is another BM with
initial data W_s

Theorem 32

Let:

- W Wiener process, with values in $\mathbb{R}^d$.
- $Y : E \rightarrow \mathbb{R}$ bounded measurable.
- $s \geq 0$.

$$\mathbb{E}_x[z]$$

Then:

$$\mathbf{E}_x[Y \circ \theta_s | \mathcal{F}_s] = \mathbf{E}_{W_s}[Y] = \mathbb{E}_x[Y]_{|x=W_s}$$

Interpretation:

Future after s can be predicted with value of W_s only.

Proof in simplest case:

Take $Y = f(W_t)$ for t fixed

Then

$$Y \circ \Theta_s = f(W_{s+t})$$

$$E[Y \circ \Theta_s | \mathcal{F}_s] = E[f(W_{s+t}) | \mathcal{F}_s]$$

$$= E[f(W_s + \delta W_{s,s+t}) | \mathcal{F}_s]$$

$$\underbrace{W_s}_{\in \mathcal{F}_s} \quad \underbrace{\delta W_{s,s+t}}_{\perp \mathcal{F}_s}$$

$$= E[f(x, \delta W_{s,s+t})]_{x=W_s}$$

Computation $\sim W(0, t \text{Id})$

$$\mathbb{E} [f (x + \sqrt{t} W_{1, s+t})]$$

$$= \int_{\mathbb{R}^d} f(z) \frac{e^{-\frac{|z-x|^2}{2t}}}{(2\pi t)^{d/2}} dz$$

↗ heat semigroup

$$= \widehat{P_t f} (x)$$

Conclusion: $\mathbb{E} [Y \circ \theta_s] = P_t f (W_s)$

$$= \mathbb{E}_{W_s} [Y]$$