

Diffusions as Markov processes

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Outline

- 1 Restarting a diffusion
- 2 Generator of a diffusion
- 3 Transition densities and Kolmogorov equations

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Setting

Data:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \tau]}, \mathbf{P})$ stochastic basis
- W d -dimensional Brownian motion
- $b, \sigma^1, \dots, \sigma^d : \mathbb{R}^n \rightarrow \mathbb{R}^n$ Lipschitz, $\alpha \in \mathbb{R}^n$

Equation: X unique solution in $L^2(\Omega; \mathcal{C}([0, \tau]))$ of

$$X_t = \alpha + \int_0^t b(X_r) dr + \sum_{k=1}^d \int_0^t \sigma^k(X_r) dW_r^k \quad (1)$$

Aim: Markov property for X , as for Brownian motion

$\hookrightarrow$ Future after time s driven by the increments of W after s

Restarted Brownian motion

Notation: For $s \in [0, \tau)$ we set

$$\begin{aligned}W_u^s &= W_{s+u} - W_s, \quad u \in [0, \tau - s] \\ \mathcal{G}^s &= \sigma \{ W_u^s; u \in [0, \tau - s] \}\end{aligned}$$

$\hookrightarrow$ Restarted filtration: $(\mathcal{F}_{s+u})_{u \in [0, \tau-s]}$

Lemma 1

Let $s \in [0, \tau)$. Then:

- W^s is a Brownian motion with respect to $(\mathcal{F}_{s+u})_{u \in [0, \tau-s]}$
- $\mathcal{G}^s$ and $\mathcal{F}_s$ are independent

Restarted diffusion

Notation: For $s \in [0, \tau)$ and $x \in \mathbb{R}^n$,
 $X^{s,x} = \{X_u^{s,x}; u \in [0, \tau - s]\}$ solution of

$$X_u^{s,x} = x + \int_0^u b(X_r^{s,x}) dr + \sum_{k=1}^d \int_0^u \sigma^k(X_r^{s,x}) dW_r^{s,k}$$

Well posedness:

- Same equation as (1), on the restarted basis, driven by W^s
- Existence and uniqueness theorem applies verbatim
- $X^{s,x}$ adapted to $(\mathcal{F}_{s+u})_{u \in [0, \tau-s]}$, and $X^{0,x} = X$

Restarting the Itô integral

Lemma 2

Let

- $v \in L^2_a([0, \tau - s])$ on the restarted basis
- $\bar{v}_r = v_{r-s} \mathbf{1}_{(s, \tau]}(r)$

Then for $u \in [0, \tau - s]$,

$$\int_0^u v_r dW_r^s = \int_s^{s+u} \bar{v}_r dW_r$$

Meaning: Integrals against W^s on $[0, u]$

- Are integrals against W on $[s, s + u]$
- Needed to restart equation (1) at time s

Proof

Formal derivation: $dW_r^s = dW_{s+r}$

$\hookrightarrow$ Then change of variable $t = s + r$:

$$\int_0^u v_r dW_{s+r} = \int_s^{s+u} v_{t-s} dW_t = \int_s^{s+u} \bar{v}_t dW_t$$

Rigorous proof: No change of variables inside an Itô integral

- Check the identity for simple processes v
 $\hookrightarrow$ finite sums of increments of W
- Pass to the limit in $L^2(\Omega)$
 $\hookrightarrow$ by density of simple processes in L_a^2

Independence from the past

Lemma 3

Let $s \in [0, \tau)$, $x \in \mathbb{R}^n$ and $u_1, \dots, u_m \in [0, \tau - s]$. Then:

$$(X_{u_1}^{s,x}, \dots, X_{u_m}^{s,x}) \perp\!\!\!\perp \mathcal{F}_s$$

In particular, for $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ bounded measurable and $u \in [0, \tau - s]$,

$$\mathbf{E}[\Phi(X_u^{s,x}) | \mathcal{F}_s] = \mathbf{E}[\Phi(X_u^{s,x})]$$

Interpretation: Analog of $W_{s+u} - W_s \perp\!\!\!\perp \mathcal{F}_s$ for the diffusion

$\hookrightarrow$ This is why the equation is restarted from a deterministic point x

Proof

Measurability: $X^{s,x}$ is a strong solution on the restarted basis
 $\hookrightarrow X_u^{s,x}$ measurable with respect to $\sigma(W_r^s; r \leq u) \subset \mathcal{G}^s$

Independence: $\mathcal{G}^s \perp\!\!\!\perp \mathcal{F}_s$, by the lemma on restarted Brownian motion
 $\hookrightarrow (X_{u_1}^{s,x}, \dots, X_{u_m}^{s,x}) \perp\!\!\!\perp \mathcal{F}_s$

Conditional expectation: For $Z \perp\!\!\!\perp \mathcal{F}_s$ integrable,

$$\mathbf{E}[Z \mid \mathcal{F}_s] = \mathbf{E}[Z]$$

$\hookrightarrow$ Apply to $Z = \Phi(X_u^{s,x})$

Restart property

Proposition 4

Let $s \in [0, \tau)$ and X solution of (1). Then for $u \in [0, \tau - s]$,

$$X_{s+u} = X_s + \int_0^u b(X_{s+r}) dr + \sum_{k=1}^d \int_0^u \sigma^k(X_{s+r}) dW_r^{s,k}$$

Interpretation: After time s , X solves the restarted equation
 $\hookrightarrow$ Started from the random point $X_s \in \mathcal{F}_s$, driven by W^s

Proof

Increment: Subtract (1) at times $s + u$ and s :

$$X_{s+u} - X_s = \int_s^{s+u} b(X_r) dr + \sum_{k=1}^d \int_s^{s+u} \sigma^k(X_r) dW_r^k$$

Lebesgue term: Change of variables $r = s + r'$, path by path

Stochastic term: Restarted Itô integral, with $v_r = \sigma^k(X_{s+r})$ and $\bar{v}_r = \sigma^k(X_r)$ on $(s, \tau]$

$$\int_s^{s+u} \sigma^k(X_r) dW_r^k = \int_0^u \sigma^k(X_{s+r}) dW_r^{s,k}$$

Transition operators

Definition 5

Let:

- $0 \leq s \leq t \leq \tau$
- $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ bounded measurable

The **transition operator** $p_{s,t}$ is defined by

$$p_{s,t}\Phi(x) = \mathbf{E} [\Phi (X_{t-s}^{s,x})], \quad x \in \mathbb{R}^n$$

Interpretation:

- Average of Φ over the solution restarted from x at time s
- Generalizes the heat semigroup of Chapter 1

Transition operators: properties

Properties:

- $p_{s,s}\Phi = \Phi$, since $X_0^{s,x} = x$
- $\|p_{s,t}\Phi\|_\infty \leq \|\Phi\|_\infty$
- Φ bounded Lipschitz $\implies p_{s,t}\Phi$ bounded Lipschitz
 $\hookrightarrow$ stability of $X^{s,x}$ in x

Brownian case: $b = 0$ and $\sigma = \text{Id}$, so that

- $X_u^{s,x} = x + W_u^s$
- $p_{s,t} = p_{t-s}$, heat semigroup of Chapter 1

Markov property

Theorem 6

Let:

- b, σ Lipschitz, X solution of (1)
- $0 \leq s \leq t \leq \tau$, and $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ bounded measurable

Then $p_{s,t}\Phi$ is measurable, and

$$\mathbf{E} [\Phi(X_t) \mid \mathcal{F}_s] = p_{s,t}\Phi(X_s)$$

Interpretation:

Future after s predicted with X_s only

$\hookrightarrow$ Solve (1) restarted from X_s , then average

Markov property: idea of proof

Step 1, X_s with countably many values x_i : Set $A_i = (X_s = x_i) \in \mathcal{F}_s$
 $\hookrightarrow Z = \sum_i \mathbf{1}_{A_i} X^{s, x_i}$ solves the restarted equation from X_s
 $\hookrightarrow Z = X_{s+}$. by uniqueness, and independence from the past gives

$$\mathbf{E} [\Phi(X_t) | \mathcal{F}_s] = \sum_i \mathbf{1}_{A_i} \mathbf{E} [\Phi(X_{t-s}^{s, x_i})] = p_{s,t} \Phi(X_s)$$

Step 2, general X_s : Discretize X_s on dyadic cubes
 $\hookrightarrow$ Stability with respect to the initial condition, for Φ Lipschitz

Step 3, general Φ : Monotone class theorem, as for Brownian motion

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Diffusion matrix

Definition 7

Let $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ the coefficients of the SDE. The **diffusion matrix** is

$$a = \sigma \sigma^* : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}, \quad \text{that is} \quad a^{j_1 j_2}(x) = \sum_{k=1}^d \sigma^{j_1 k}(x) \sigma^{j_2 k}(x)$$

Properties:

- $a(x)$ symmetric and nonnegative, since $\langle a(x)y, y \rangle = |\sigma(x)^* y|^2$
- a Lipschitz on bounded sets whenever σ is Lipschitz

Generator

Definition 8

Let:

- b, σ coefficients of the SDE, $a = \sigma \sigma^*$ the diffusion matrix
- $f \in \mathcal{C}^2(\mathbb{R}^n)$

The **generator** $\mathcal{L}$ of the diffusion is defined by

$$\mathcal{L}f(x) = \sum_{j=1}^n b^j(x) \partial_{x_j} f(x) + \frac{1}{2} \sum_{j_1, j_2=1}^n a^{j_1 j_2}(x) \partial_{x_{j_1} x_{j_2}}^2 f(x)$$

Interpretation: Second order differential operator

↪ First order part from the drift, second order part from the noise

Generator: examples

Brownian motion: $b = 0$, $\sigma = \text{Id}$, hence $a = \text{Id}$ and

$$\mathcal{L}f = \frac{1}{2} \sum_{j=1}^n \partial_{x_j x_j}^2 f = \frac{1}{2} \Delta f$$

$\hookrightarrow$ Operator of the heat equation seen in Chapter 1

Case $n = d = 1$: $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$, hence $a = \sigma^2$ and

$$\mathcal{L}f = b f' + \frac{1}{2} \sigma^2 f''$$

$\hookrightarrow$ Setting of the proof below

Integrated generator identity

Lemma 9

Let b, σ Lipschitz, $s \in [0, \tau)$, $x \in \mathbb{R}^n$ and $f \in \mathcal{C}_b^2(\mathbb{R}^n)$. Then for $u \in [0, \tau - s]$,

$$p_{s,s+u}f(x) = f(x) + \int_0^u \mathbf{E}[\mathcal{L}f(X_r^{s,x})] dr$$

Interpretation: Evolution of $\mathbf{E}[f(X_u^{s,x})]$ governed by $\mathcal{L}$

$\hookrightarrow$ Analog of the heat semigroup identity for Brownian motion

Proof: strategy

Two steps:

- $X^{s,x}$ is an Itô process on the restarted basis
- Itô's formula for $f(X_u^{s,x})$, then take expectations

Below: Proof for $n = d = 1$, after recalling the Itô process slides
↪ General case: same computation with indices

Recall: Itô process in dimension 1

Definition 10

Let:

- $\alpha \in \mathbb{R}$, and b bounded adapted real process
- σ real process in $L^2_a([0, \tau])$

We say that X is an **Itô process** if

$$X_t = \alpha + \int_0^t b_r dr + \int_0^t \sigma_r dW_r$$

Solution of (1) with $n = d = 1$: Itô process with

$\hookrightarrow b_r = b(X_r)$ and $\sigma_r = \sigma(X_r)$

$\hookrightarrow b(X_\cdot)$ only continuous, which is enough for $\int_0^t b_r dr$

Recall: Itô's formula in dimension 1

Proposition 11

Let X Itô process defined by α, b, σ , and $f \in \mathcal{C}_b^2(\mathbb{R})$. Then:

$$\begin{aligned} f(X_t) &= f(\alpha) + \int_0^t f'(X_r) b_r dr + \int_0^t f'(X_r) \sigma_r dW_r \\ &\quad + \frac{1}{2} \int_0^t f''(X_r) \sigma_r^2 dr \end{aligned}$$

Remark: $\int_0^t f'(X_r) \sigma_r dW_r$ is a martingale

↪ Since f' bounded and $\sigma \in L_a^2([0, \tau])$

↪ Vanishing expectation

Proof of the integrated identity for $n = d = 1$

Itô process: On the restarted basis, with $b_r = b(X_r^{s,x})$ and $\sigma_r = \sigma(X_r^{s,x})$

Itô's formula: Grouping the dr terms with $\mathcal{L}f = b f' + \frac{1}{2}\sigma^2 f''$,

$$f(X_u^{s,x}) = f(x) + \int_0^u \mathcal{L}f(X_r^{s,x}) dr + \int_0^u f'(X_r^{s,x}) \sigma(X_r^{s,x}) dW_r^s$$

Expectation:

- Stochastic integral has mean 0, and $\mathbf{E}[f(X_u^{s,x})] = p_{s,s+u}f(x)$
- Fubini for the dr term: $|\mathcal{L}f(z)| \leq c_f(1 + |z|^2)$ and $\mathbf{E}[\sup_r |X_r^{s,x}|^2] < \infty$

Generator of a diffusion

Theorem 12

Let b, σ Lipschitz, $s \in [0, \tau)$ and $f \in \mathcal{C}_b^2(\mathbb{R}^n)$. Then for all $x \in \mathbb{R}^n$,

$$\lim_{h \rightarrow 0^+} \frac{p_{s,s+h} f(x) - f(x)}{h} = \mathcal{L}f(x)$$

Interpretation: $\mathcal{L}$ is the infinitesimal generator of

$\{p_{s,t}; 0 \leq s \leq t \leq \tau\}$

$\hookrightarrow$ Deterministic operator, coefficients depend on the current position only

Brownian case: $p_{s,s+h} = p_h$ heat semigroup and $\mathcal{L} = \frac{1}{2}\Delta$

$\hookrightarrow$ Generator of Brownian motion, seen in Chapter 2

Proof

Step 1, continuity at $r = 0$: We have

$$\lim_{r \rightarrow 0^+} \mathbf{E}[\mathcal{L}f(X_r^{s,x})] = \mathcal{L}f(x)$$

- $X_r^{s,x} \rightarrow x$ a.s. (continuous paths, $X_0^{s,x} = x$), and $\mathcal{L}f$ continuous
- Domination: $|\mathcal{L}f(X_r^{s,x})| \leq c_f(1 + \sup_r |X_r^{s,x}|^2)$, integrable

Step 2, averages: Integrated identity divided by h :

$$\frac{p_{s,s+h}f(x) - f(x)}{h} = \frac{1}{h} \int_0^h \mathbf{E}[\mathcal{L}f(X_r^{s,x})] dr \xrightarrow{h \rightarrow 0^+} \mathcal{L}f(x)$$

$\hookrightarrow$ Average of a bounded function, continuous at $r = 0$

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Additive noise

Setting: $d = n$, and the equation with **additive noise**

$$X_t = X_0 + \int_0^t b(r, X_r) dr + \int_0^t \sigma(r) dW_r \quad (2)$$

- $b : [0, \tau] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : [0, \tau] \rightarrow \mathbb{R}^{n \times n}$
- $X_0 \in \mathcal{F}_0$ random initial condition, with law μ

Comments:

- Well posed path by path (Chapter 3)
- Forward equation of generative models, reversed in Chapter 5

Smoothness assumption

Assumption:

- 1 b infinitely differentiable, bounded, with bounded derivatives of all orders
- 2 σ continuously differentiable, and $\sigma(t)$ invertible for all $t \in [0, \tau]$

Remarks:

- Far from minimal, chosen for convenience
- In force for the rest of the deck

Well posedness: b Lipschitz in x , uniformly in t

$\hookrightarrow$ Existence and uniqueness for (2)

Uniform ellipticity

Lemma 13

Let σ as in the smoothness assumption, and $a(t) = \sigma(t)\sigma(t)^*$. Then a is bounded on $[0, \tau]$, and there is $\lambda > 0$ with

$$\langle a(t)y, y \rangle = |\sigma(t)^*y|^2 \geq \lambda |y|^2, \quad t \in [0, \tau], \quad y \in \mathbb{R}^n$$

Interpretation:

- Noise acts in every direction, with strength bounded below
- Hypothesis of the parabolic theory giving smooth densities

Proof

Compactness argument:

- $(t, y) \mapsto |\sigma(t)^* y|^2$ continuous on the compact set $K = [0, \tau] \times \{|y| = 1\}$
- Positive on K , since $\sigma(t)^*$ invertible and $y \neq 0$
- Hence a positive minimum λ on K

Conclusion: For $y \neq 0$, by homogeneity,

$$|\sigma(t)^* y|^2 = |y|^2 \left| \sigma(t)^* \frac{y}{|y|} \right|^2 \geq \lambda |y|^2$$

$\Leftrightarrow$ Boundedness of a by continuity of σ on $[0, \tau]$

Restarted process in the additive case

Restarted process: For $s \in [0, \tau)$ and $x \in \mathbb{R}^n$, $Y^{s,x}$ solution of

$$Y_u^{s,x} = x + \int_0^u b(s+r, Y_r^{s,x}) dr + \int_0^u \sigma(s+r) dW_r^s, \quad u \in [0, \tau-s]$$

$\hookrightarrow$ Transition operator $\tilde{p}_{s,t} \Phi(x) = \mathbf{E} [\Phi(Y_{t-s}^{s,x})]$

Markov property: For X solution of (2), with random X_0 ,

$$\mathbf{E} [\Phi(X_t) | \mathcal{F}_s] = \tilde{p}_{s,t} \Phi(X_s)$$

$\hookrightarrow$ Same proof as in the Lipschitz case, restarted equation well posed

Generator in the additive case

Generator: Time dependent operator, for $\varphi \in \mathcal{C}^2(\mathbb{R}^n)$,

$$\mathcal{L}_s \varphi(x) = \sum_{j=1}^n b^j(s, x) \partial_{x_j} \varphi(x) + \frac{1}{2} \sum_{j_1, j_2=1}^n a^{j_1 j_2}(s) \partial_{x_{j_1} x_{j_2}}^2 \varphi(x)$$

Integrated identity: Same proof as before, for $\varphi \in \mathcal{C}_b^2(\mathbb{R}^n)$

$$\tilde{p}_{s,s+u} \varphi(x) = \varphi(x) + \int_0^u \mathbf{E} [\mathcal{L}_{s+r} \varphi(Y_r^{s,x})] dr$$

$\hookrightarrow$ Two time variables: $s + r$ for the operator, r for the process

Adjoint operator

Definition 14

Let:

- b, σ satisfying the smoothness assumption, $a(t) = \sigma(t)\sigma(t)^*$
- $t \in [0, \tau]$ and $\psi \in \mathcal{C}^2(\mathbb{R}^n)$

The **adjoint operator** $\mathcal{L}_t^*$ is defined by

$$\mathcal{L}_t^* \psi(x) = - \sum_{j=1}^n \partial_{x_j} [b^j(t, x) \psi(x)] + \frac{1}{2} \sum_{j_1, j_2=1}^n a^{j_1 j_2}(t) \partial_{x_{j_1} x_{j_2}}^2 \psi(x)$$

Interpretation: Formal adjoint of $\mathcal{L}_t$ in $L^2(\mathbb{R}^n)$ (see duality below)

Adjoint operator: remarks

Comparison with $\mathcal{L}_t$:

- Only the first order term changes with respect to $\mathcal{L}_t$
- Second order term unchanged, since $a(t)$ does not depend on x
 $\hookrightarrow$ With $\sigma(t, x)$, one gets $\frac{1}{2} \partial_{x_{j_1} x_{j_2}}^2 [a^{ij_2}(t, x) \psi(x)]$ instead

Role: $\mathcal{L}_t^*$ governs the evolution of densities (see below)

Duality

Lemma 15

Let $t \in [0, \tau]$, $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\psi \in \mathcal{C}^2(\mathbb{R}^n)$. Then:

$$\int_{\mathbb{R}^n} \mathcal{L}_t \varphi(x) \psi(x) dx = \int_{\mathbb{R}^n} \varphi(x) \mathcal{L}_t^* \psi(x) dx$$

Comments:

- Justifies the name adjoint
- Only property of $\mathcal{L}_t^*$ used in the sequel
- Integrals over a ball containing $\text{Supp}(\varphi)$, no convergence issue

Proof

First order term: One integration by parts, no boundary term (φ compactly supported)

$$\int_{\mathbb{R}^n} b^j(t, x) \partial_{x_j} \varphi(x) \psi(x) dx = - \int_{\mathbb{R}^n} \varphi(x) \partial_{x_j} [b^j(t, x) \psi(x)] dx$$

Second order term: Two integrations by parts, $a^{j_1 j_2}(t)$ constant in x

$$\int_{\mathbb{R}^n} a^{j_1 j_2}(t) \partial_{x_{j_1} x_{j_2}}^2 \varphi(x) \psi(x) dx = \int_{\mathbb{R}^n} a^{j_1 j_2}(t) \varphi(x) \partial_{x_{j_1} x_{j_2}}^2 \psi(x) dx$$

Conclusion: Sum over j , then over j_1, j_2

$\hookrightarrow$ Definition of $\mathcal{L}_t^*$ under the integral sign

Chapman–Kolmogorov equation

Theorem 16

Let b, σ satisfying the smoothness assumption, $0 \leq s \leq t \leq t' \leq \tau$ and $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ bounded measurable. Then:

$$\tilde{p}_{s,t'} \Phi = \tilde{p}_{s,t} (\tilde{p}_{t,t'} \Phi)$$

Interpretation: Predict over $[s, t]$, then over $[t, t']$

$\hookrightarrow$ Same as predicting over $[s, t']$ directly

Remarks:

- Brownian case: semigroup property $p_{t'-s} = p_{t-s} \circ p_{t'-t}$
- Same statement and proof for $p_{s,t}$ in the Lipschitz case

Proof

Markov property on the restarted basis: For $Y^{s,x}$, at times $t - s \leq t' - s$,

$$\mathbf{E} \left[\Phi(Y_{t'-s}^{s,x}) \mid \mathcal{F}_t \right] = \tilde{p}_{t,t'} \Phi(Y_{t-s}^{s,x})$$

$\hookrightarrow$ Restarting $Y^{s,x}$ at time $t - s$ is restarting X at time t
Shifted coefficients $b(t + r, \cdot)$ and $\sigma(t + r)$

Tower property:

$$\tilde{p}_{s,t'} \Phi(x) = \mathbf{E} \left[\Phi(Y_{t'-s}^{s,x}) \right] = \mathbf{E} \left[\mathbf{E} \left[\Phi(Y_{t'-s}^{s,x}) \mid \mathcal{F}_t \right] \right] = \mathbf{E} \left[\tilde{p}_{t,t'} \Phi(Y_{t-s}^{s,x}) \right]$$

$\hookrightarrow$ Right hand side is $\tilde{p}_{s,t}(\tilde{p}_{t,t'} \Phi)(x)$, by definition of $\tilde{p}_{s,t}$

Transition density

Theorem 17

Under the smoothness assumption, there is a function $\tilde{q}_{s,t}(x, y)$, for $0 \leq s < t \leq \tau$ and $x, y \in \mathbb{R}^n$, such that:

- $\tilde{q}_{s,t}(x, \cdot)$ is the density of $Y_{t-s}^{s,x}$, that is

$$\tilde{p}_{s,t}\Phi(x) = \int_{\mathbb{R}^n} \tilde{q}_{s,t}(x, y) \Phi(y) dy$$

- $\tilde{q}$ is infinitely differentiable in (s, t, x, y) , and $\tilde{q}_{s,t}(x, y) > 0$

Comments:

- Admitted: parabolic PDE theory (Friedman, Durrett Section 7.3)
- Fundamental solution of the Kolmogorov equations (next slide)

Kolmogorov equations

Theorem 18

Under the smoothness assumption, $\tilde{q}$ satisfies:

- *Backward equation, (t, y) fixed, $\mathcal{L}_s$ acts on x :*

$$-\partial_s \tilde{q}_{s,t}(x, y) = \mathcal{L}_s \tilde{q}_{s,t}(\cdot, y)(x)$$

- *Forward (Fokker–Planck) equation, (s, x) fixed, $\mathcal{L}_t^*$ acts on y :*

$$\partial_t \tilde{q}_{s,t}(x, y) = \mathcal{L}_t^* \tilde{q}_{s,t}(x, \cdot)(y)$$

Dirac condition: $\tilde{q}_{s,t}(x, y) dy \rightarrow \delta_x$ weakly as $t - s \rightarrow 0$

$\hookrightarrow$ Continuity of the paths of $Y^{s,x}$

Backward versus forward

Backward equation:

- Starting point (s, x) moves, arrival point (t, y) frozen
- Operator $\mathcal{L}_s$, produced by Itô's formula

Forward equation:

- Arrival point (t, y) moves, starting point (s, x) frozen
- Adjoint operator $\mathcal{L}_t^*$, describes the transport of mass

In the sequel: Forward equation is the relevant one

↪ Evolution of the law of X_t , and the reversal of Chapter 5

Chapman–Kolmogorov for densities

Corollary 19

Let $0 \leq s < r < t \leq \tau$ and $x \in \mathbb{R}^n$. Then for a.e. $y \in \mathbb{R}^n$,

$$\tilde{q}_{s,t}(x, y) = \int_{\mathbb{R}^n} \tilde{q}_{s,r}(x, z) \tilde{q}_{r,t}(z, y) dz$$

Interpretation: Chapman–Kolmogorov at the level of densities
 $\hookrightarrow$ Go from x to y through an intermediate point z at time r

Remark: Holds for every y , by continuity of $\tilde{q}$ (not needed below)

Proof

Operators: Chapman–Kolmogorov $\tilde{p}_{s,t}\Phi = \tilde{p}_{s,r}(\tilde{p}_{r,t}\Phi)$, written with densities

$$\int_{\mathbb{R}^n} \tilde{q}_{s,t}(x, y) \Phi(y) dy = \int_{\mathbb{R}^n} \tilde{q}_{s,r}(x, z) \left(\int_{\mathbb{R}^n} \tilde{q}_{r,t}(z, y) \Phi(y) dy \right) dz$$

Fubini: Exchange dy and dz (absolutely convergent, Φ bounded)

$$\int_{\mathbb{R}^n} \tilde{q}_{s,t}(x, y) \Phi(y) dy = \int_{\mathbb{R}^n} \Phi(y) \left(\int_{\mathbb{R}^n} \tilde{q}_{s,r}(x, z) \tilde{q}_{r,t}(z, y) dz \right) dy$$

$\hookrightarrow \Phi$ arbitrary, so both integrands in y coincide a.e.

Marginal density

Notation: X solution of (2) with $X_0 \sim \mu$. For $t \in (0, \tau]$,

$$p_t(x) = \int_{\mathbb{R}^n} \tilde{q}_{0,t}(z, x) \mu(dz), \quad x \in \mathbb{R}^n$$

$\hookrightarrow \mu$ arbitrary (e.g. empirical measure of a data set)

Proposition 20

Under the smoothness assumption:

- p_t is the density of X_t
- p infinitely differentiable on $(0, \tau] \times \mathbb{R}^n$, and $p_t(x) > 0$
- Fokker–Planck equation: $\partial_t p_t(x) = \mathcal{L}_t^* p_t(x)$

Weak form of the forward equation

Proposition 21

Let b Lipschitz in x only, X solution of (2), and ν_t law of X_t . Then for $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $0 \leq s \leq t \leq \tau$,

$$\int_{\mathbb{R}^n} \varphi d\nu_t = \int_{\mathbb{R}^n} \varphi d\nu_s + \int_s^t \left(\int_{\mathbb{R}^n} \mathcal{L}_r \varphi d\nu_r \right) dr$$

Comments:

- Itô's formula alone, no parabolic theory, no density needed
- Fokker–Planck equation in weak form, tested against φ

Proof (1)

Integrated identity: For $z \in \mathbb{R}^n$, with $r = s + u$ and r frozen in $\mathcal{L}_r \varphi$,

$$\tilde{p}_{s,t} \varphi(z) = \varphi(z) + \int_s^t \mathbf{E} [\mathcal{L}_r \varphi(Y_{r-s}^{s,z})] dr = \varphi(z) + \int_s^t \tilde{p}_{s,r} (\mathcal{L}_r \varphi)(z) dr$$

Markov property: For Ψ bounded measurable and $r \in [s, \tau]$,

$$\int_{\mathbb{R}^n} \tilde{p}_{s,r} \Psi d\nu_s = \mathbf{E} [\tilde{p}_{s,r} \Psi(X_s)] = \mathbf{E} [\Psi(X_r)] = \int_{\mathbb{R}^n} \Psi d\nu_r$$

Next: Integrate the first identity in $\nu_s(dz)$, then apply the second

Proof (2)

Integration in $\nu_s(dz)$: Then Fubini, since the integrand is bounded

$$\int_{\mathbb{R}^n} \tilde{p}_{s,t} \varphi d\nu_s = \int_{\mathbb{R}^n} \varphi d\nu_s + \int_s^t \left(\int_{\mathbb{R}^n} \tilde{p}_{s,r} (\mathcal{L}_r \varphi) d\nu_s \right) dr$$

Markov identity twice:

- $\Psi = \varphi$ at time t : left hand side is $\int_{\mathbb{R}^n} \varphi d\nu_t$
- $\Psi = \mathcal{L}_r \varphi$ at time r (bounded, φ compactly supported): inner integral is $\int_{\mathbb{R}^n} \mathcal{L}_r \varphi d\nu_r$

Conclusion:

$$\int_{\mathbb{R}^n} \varphi d\nu_t = \int_{\mathbb{R}^n} \varphi d\nu_s + \int_s^t \left(\int_{\mathbb{R}^n} \mathcal{L}_r \varphi d\nu_r \right) dr$$

From the weak form to Fokker–Planck

Density of X_t : Markov property at $s = 0$, then Fubini

$$\mathbf{E}[\Phi(X_t)] = \int_{\mathbb{R}^n} \tilde{p}_{0,t} \Phi(z) \mu(dz) = \int_{\mathbb{R}^n} \Phi(x) \left(\int_{\mathbb{R}^n} \tilde{q}_{0,t}(z, x) \mu(dz) \right) dx$$

$\hookrightarrow \nu_t(dx) = p_t(x) dx$, with $p_t \in \mathcal{C}^2$ by the parabolic theory

Duality: The weak form becomes

$$\int_{\mathbb{R}^n} \varphi(x) \left[p_t(x) - p_s(x) - \int_s^t \mathcal{L}_r^* p_r(x) dr \right] dx = 0$$

$\hookrightarrow \varphi$ arbitrary, bracket continuous, hence $= 0$. Differentiate in t

Summary

Markov structure of a diffusion:

- Transition operators $p_{s,t}$, Markov property, Chapman–Kolmogorov
- Generator $\mathcal{L}$, obtained from Itô's formula

Additive noise, smooth coefficients:

- Transition density $\tilde{q}$, backward and forward equations
- Density p_t of X_t , Fokker–Planck equation $\partial_t p_t = \mathcal{L}_t^* p_t$

Next: Run X backward in time

↪ Reversed process is a diffusion, drift corrected by the score

$$\nabla \log p_t$$