Probability preliminaries

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Stochastic calculus - MA598
Outline

1. Basic structures
2. Products of probability spaces
3. Gaussian random vectors
4. Conditional expectation
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1. Basic structures

2. Products of probability spaces

3. Gaussian random vectors

4. Conditional expectation
First definitions

Probability space: \((\Omega, \mathcal{F}, P)\) with
- \(\Omega\) set
- \(\mathcal{F}\) generic \(\sigma\)-algebra
- \(P\) probability measure

Hypothesis: We assume that \(P\) is complete, i.e

\[ A \in \mathcal{F} \text{ such that } P(A) = 0, \quad \text{and } B \subset A \implies B \in \mathcal{F} \text{ and } P(B) = 0 \]

Remark: One can always complete a probability space
Simple examples

Rolling 2 dice:
- $\Omega = \{1, 2, 3, 4, 5, 6\}^2$
- $\mathcal{F} = \mathcal{P}(\Omega)$
- $P(A) = \frac{|A|}{36}$

Uniform law on $[0, 1]$:
- $\Omega = [0, 1]$
- $\mathcal{F} = \mathcal{B}([0, 1])$
- $P = \lambda$, Lebesgue measure

Gaussian law on $\mathbb{R}$:
- $\Omega = \mathbb{R}$
- $\mathcal{F} = \mathcal{B}(\mathbb{R})$
- $P(A) = \frac{1}{(2\pi)^{1/2}} \int_A e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$, for $A \in \mathcal{F}$
Important example for stochastic processes

**Proposition 1.**

Let $\Omega = \mathcal{C}([0, \infty); \mathbb{R}^m)$. We set:

$$d(f, g) = \sum_{n \geq 1} \frac{\|f - g\|_{\infty,n}}{2^n (1 + \|f - g\|_{\infty,n})},$$

where

$$\|f - g\|_{\infty,n} = \sup \{ |f_t - g_t|; \ t \in [0, n] \}.$$

Then $\Omega$ is a complete separable metric space.

Following chapter: We construct the Wiener measure on $\Omega$
Independence

Independence of r.v: Let $(X_j)_{j \in J}$ r.v, with values in $\mathbb{R}^n$. We say that these r.v are independent if for all $m \geq 2$:

- For all $j_1, \ldots, j_m \in J$, the r.v $(X_{j_1}, \ldots, X_{j_m})$ are $\perp \perp$
- Otherwise stated: for all $A_1, \ldots, A_m \in \mathcal{B}(\mathbb{R}^n)$ we have

$$
P \left( X_{j_1} \in A_1, \ldots, X_{j_m} \in A_m \right) = \prod_{k=1}^{m} P \left( X_{j_k} \in A_k \right)
$$

Independence of $\sigma$-algebras: Let $(\mathcal{F}_j)_{j \in J}$ $\sigma$-algebras, $\mathcal{F}_j \subset \mathcal{F}$. We say that those $\sigma$-algebras are independent if for all $m \geq 2$:

- For all $j_1, \ldots, j_m \in J$, the $\sigma$-algebras $(\mathcal{F}_{j_1}, \ldots, \mathcal{F}_{j_m})$ are $\perp \perp$
- Otherwise stated: for all $B_1 \in \mathcal{F}_{j_1}, \ldots, B_m \in \mathcal{F}_{j_m}$ we have

$$
P \left( \bigcap_{k=1}^{m} B_k \right) = \prod_{k=1}^{m} P \left( B_k \right)
$$
\section*{π-systems and λ-systems}

\textbf{π-system:} Let $\mathcal{P}$ family of sets in $\Omega$. $\mathcal{P}$ is a $\pi$-system if:

\[ A, B \in \mathcal{P} \implies A \cap B \in \mathcal{P} \]

\textbf{λ-system:} Let $\mathcal{L}$ family of sets in $\Omega$. $\mathcal{L}$ is a $\lambda$-system if:

1. $\Omega \in \mathcal{L}$
2. If $A \in \mathcal{L}$, then $A^c \in \mathcal{L}$
3. If for $j \geq 1$ we have:
   - $A_j \in \mathcal{L}$
   - $A_j \cap A_i = \emptyset$ if $j \neq i$

Then $\bigcup_{j \geq 1} A_j \in \mathcal{L}$
Dynkin’s $\pi$-$\lambda$ lemma

Lemma 2.

Let $\mathcal{P}$ and $\mathcal{L}$ such that:

- $\mathcal{P}$ is a $\pi$-system
- $\mathcal{L}$ is a $\lambda$-system
- $\mathcal{P} \subset \mathcal{L}$

Then $\sigma(\mathcal{P}) \subset \mathcal{L}$
Application of Dynkin’s lemma

Proposition 3.

Let:
- \( X_1, \ldots, X_n \) r.v, with values in \( \mathbb{R}^m \).
- \( X \equiv (X_1, \ldots, X_n) \in \mathbb{R}^{m \times n} \).
- \( \mu_{X_j} = \mathcal{L}(X_j) \) and \( \mu_X = \mathcal{L}(X) \).

Then the following two statements are equivalent:

1. \( X_1, \ldots, X_n \) are independent
2. \( \mu_X = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n} \) on \( \mathcal{B}(\mathbb{R}^{m \times n}) \)
Definition of two systems: We set

\[ \mu_1 = \mu_X, \quad \text{and} \quad \mu_2 = \mu_X \otimes \cdots \otimes \mu_X, \]

and

\[ \mathcal{P} \equiv \left\{ A \in \mathcal{B}(\mathbb{R}^{m \times n}); A = A_1 \times \cdots \times A_n, \text{ where } A_j \in \mathcal{B}(\mathbb{R}^m) \right\} \]

\[ \mathcal{L} \equiv \left\{ B \in \mathcal{B}(\mathbb{R}^{m \times n}); \mu_1(B) = \mu_2(B) \right\}. \]

Application of Dynkin’s lemma: We have

- \( \mathcal{P} \) is a \( \pi \)-system
- \( \mathcal{L} \) is a \( \lambda \)-system
- \( \mu_1(C) = \mu_2(C) \) for all \( C \in \mathcal{P} \)

Thus \( \sigma(\mathcal{P}) \subset \mathcal{L} \), and \( \sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R}^{m \times n}) \)
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Situation: We consider \( \{ \mu_k; \ k \geq 1 \} \) sequence of probability measures on \( \mathbb{R} \).

We try to define:

\[
\mu = \bigotimes_{k=1}^{\infty} \mu_k,
\]

on a probability space

\[
\Omega \equiv \prod_{k=1}^{\infty} \Omega_k.
\]
Cylinder sets

Recall: We consider

- $(\Omega_k, \mathcal{F}_k, P_k)$ family of probability spaces
- $\Omega \equiv \prod_{k=1}^{\infty} \Omega_k$

**Definition 4.**

Let $A \subset \Omega$. We say that $A$ is cylindrical if there exists $k \geq 0$ and $0 \leq n_1 < \cdots < n_k$ such that

$$A = \{ \omega \in \Omega; \; \omega_{n_1} \in A_1, \ldots, \omega_{n_k} \in A_k \}, \quad \text{where} \quad A_j \in \mathcal{F}_{n_j}$$

Interpretation:

A cylindrical set only involves a finite number of coordinates product $\sigma$-algebra on $\Omega$: $\mathcal{F} \equiv \sigma(\mathcal{C})$, with $\mathcal{C} \equiv$ cylindrical sets.
Product measure

Theorem 5.

Let:
- \((\Omega_k, \mathcal{F}_k, P_k)\) family of probability spaces
- \((\Omega, \mathcal{F})\) product space

Then there exists a unique probability \(P\) on \((\Omega, \mathcal{F})\) such that:

\[
P(A) = \prod_{j=1}^{k} P_{n_j}(A_j), \quad \text{for all} \quad A \in \mathcal{C}
\]
Sequence of independent random variables

Theorem 6.

Let:

- \( \{\mu_k; k \geq 1\} \) family of probability laws on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\).

Then there exists:

- Probability space \((\Omega, \mathcal{F}, P)\)
- \( \{X_k; k \geq 1\} \) family of independent r.v defined on \(\Omega\)

Such that \(\mathcal{L}(X_k) = \mu_k\).
Proof

Product space: We consider

- \((\Omega_k, \mathcal{F}_k, P_k) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_k)\)
- \((\Omega, \mathcal{F}, P) \equiv \text{product space}\)
- \(X_k(\omega) = \omega_k\) if \(\omega = (\omega_k)_{k \geq 1}\)

Independence: For all \(k_1 < \ldots < k_n\) the r.v \(X_{k_j}\) are \(\perp\perp\). Indeed,

\[
P\left( \bigcap_{j=1}^{n} (X_{k_j} \in A_j) \right) = \prod_{j=1}^{n} \mu_{k_j}(A_j) = \prod_{j=1}^{n} P(X_{k_j} \in A_j),
\]

This corresponds to the definition of independence.
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**Definition**

**Definition:** Let $X \in \mathbb{R}^n$. 
$X$ is a Gaussian random vector(s) if for all $\lambda \in \mathbb{R}^n$

$$\langle \lambda, X \rangle = \lambda^* X = \sum_{i=1}^{n} \lambda_i X_i$$
is a real valued Gaussian r.v.

**Remarks:**

1. $X$ Gaussian vector
   $\Rightarrow$ Each component $X_i$ of $X$ is Gaussian real r.v.

2. Key example of Gaussian vector:
   Independent components $X_1, \ldots, X_n$

3. One can easily construct an example of $X \in \mathbb{R}^2$ such that
   (i) $X_1, X_2$ real Gaussian (ii) $X$ is not a Gaussian vector
Characteristic function

Proposition 7.

Let $X$ Gaussian vector, with mean $m$ and covariance $K$. Then, for all $u \in \mathbb{R}^n$,

$$E[\exp(\imath\langle u, X \rangle)] = e^{\imath\langle u, m \rangle - \frac{1}{2} u^* Ku},$$

where $u$ is understood as a matrix.
Proof

Random variable $\langle u, X \rangle$:
$\langle u, X \rangle$ Gaussian r.v. by assumption, with

\[
\mu := \mathbb{E}[\langle u, X \rangle] = \langle u, m \rangle, \quad \text{and} \quad \sigma^2 := \text{Var}(\langle u, X \rangle) = u^* Ku.
\]

Recall: let $Y \sim \mathcal{N}(\mu, \sigma^2)$. Then

\[
\mathbb{E}[\exp(\imath t Y)] = \exp \left( \imath t \mu - \frac{t^2}{2} \sigma^2 \right), \quad t \in \mathbb{R}.
\]
Gaussian moments

Proposition 8.

Let $X \sim \mathcal{N}(0, 1)$. Then for all $n \in \mathbb{N}$, we have:

$$E[X^n] = \begin{cases} 
0 & \text{if } n \text{ odd,} \\
\frac{(2m)!}{m!2^m} & \text{if } n \text{ even, } n = 2m.
\end{cases}$$
Affine transformations

**Notation:** If $X$ Gaussian vector with mean $m$ and covariance $K$
We write $X \sim \mathcal{N}(m, K)$

**Proposition 9.**

Let $X \sim \mathcal{N}(m_X, K_X)$, $A \in \mathbb{R}^{p,n}$ and $z \in \mathbb{R}^p$.
We set $Y = AX + z$. Then

$$Y \sim \mathcal{N}(m_Y, K_Y), \quad \text{with} \quad m_Y = z + Am_X, \quad K_Y = AK_XA^*.$$
Gaussian density

**Theorem 10.**

Let \( X \sim \mathcal{N}(m, K) \). Then

1. \( X \) admits a density iff \( K \) is invertible.
2. If \( K \) is invertible, the density of \( X \) is given by:

\[
f(x) = \frac{1}{(2\pi)^{n/2}(\det(K))^{1/2}} \exp \left( -\frac{1}{2}(x - m)^* K^{-1} (x - m) \right).
\]
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Formal definition

**Definition 11.**

A probability space \((\Omega, \mathcal{F}, P)\) is given and
- A \(\sigma\)-algebra \(\mathcal{G} \subset \mathcal{F}\).
- \(X \in \mathcal{F}\) such that \(E[|X|] < \infty\).

Conditional expectation of \(X\) given \(\mathcal{G}\):
- Denoted by: \(E[X|\mathcal{G}]\)
- Defined by: \(E[X|\mathcal{G}]\) is the r.v \(Y \in L^1(\Omega)\) such that
  1. \(Y \in \mathcal{G}\).
  2. For all \(A \in \mathcal{G}\), we have
     \[ E[X 1_A] = E[Y 1_A]. \]
Easy examples

Example 1: If $X \in \mathcal{F}$, then $\mathbf{E}[X|\mathcal{F}] = X$.

Definition: We say that $X \perp \mathcal{F}$ if $\sigma(X) \perp \mathcal{F}$

$\iff$ for all $A \in \mathcal{F}$ and $B \in \mathcal{B}(\mathbb{R})$, we have

$$P((X \in B) \cap A) = P(X \in B)P(A),$$

or otherwise stated $X \perp 1_A$.

Example 2: If $X \perp \mathcal{F}$, then $\mathbf{E}[X|\mathcal{F}] = \mathbf{E}[X]$. 