Lipschitz geometry of definable surface germs

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All sets and maps are definable in a polynomially bounded o-minimal structure over $\mathbb{R}$ with the field of exponents $\mathbb{F}$, e.g., semialgebraic or subanalytic with $\mathbb{F} = \mathbb{Q}$.

A set $X \subset \mathbb{R}^n$ inherits from $\mathbb{R}^n$ two metrics:
the **outer metric** $dist(x, y) = |y - x|$ and the **inner metric** $idist(x, y) =$ length of the shortest path in X connecting x and y.

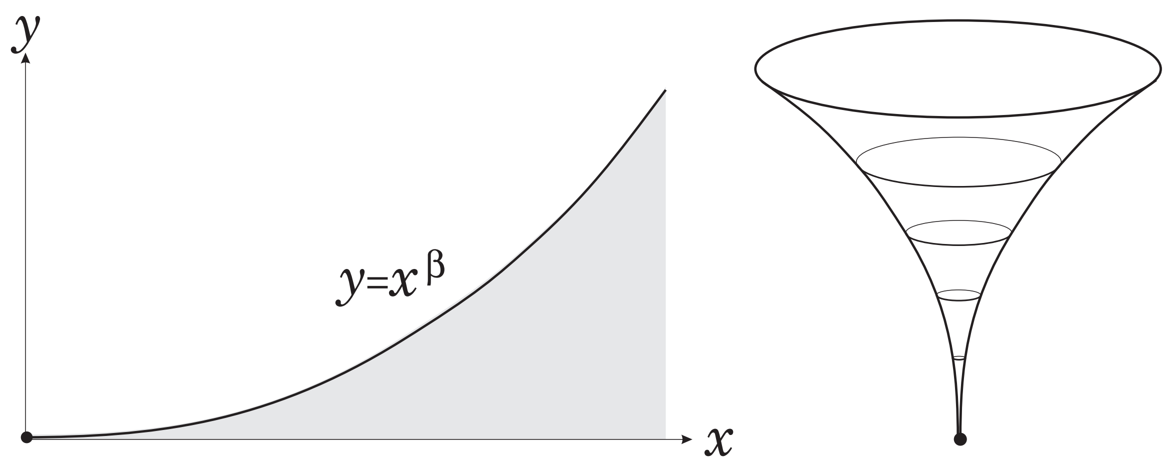
X is **normally embedded** if these two metrics on X are equivalent.

A **surface germ** is a closed two-dimensional germ X at the origin.

Germs X and Y are **outer (inner) Lipschitz equivalent** if there is an outer (inner) bi-Lipschitz homeomorphism $X \to Y$.

For $\beta \in \mathbb{F}$, $\beta \geq 1$, the **standard β-Hölder triangle** is the set $T_\beta = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ 0 \leq y \leq x^\beta\}$.

The **standard β-horn** is $C_\beta = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0,\ x^2 + y^2 = z^{2\beta}\}$.



A **β-Hölder triangle** is a germ inner Lipschitz equivalent to T_β.

A **β-horn** is a germ inner Lipschitz equivalent to C_β.

Inner Lipschitz classification of surface germs: Birbrair 99.
Canonical decomposition of a surface germ X into β_i-Hölder triangles with singular boundary arcs and β_j-horns.
Complete invariant of the inner Lipschitz equivalence class of X.

Finiteness theorems: Mostowski 85, Parusinski 94, Valette 05.
Any definable family has finitely many outer Lipschitz equivalence classes.

Our goal: Outer Lipschitz classification of surface germs.
Decomposition of X into normally embedded Hölder triangles, with some additional data, unique up to outer Lipschitz equivalence.
Complete invariant of the outer Lipschitz equivalence class of X.

An **arc** γ in X is a germ of a map $\gamma : [0, \epsilon) \to X$ such that $|\gamma(t)| = t$.

The **Valette link** $V(X)$ is the space of all arcs in X.

Tangency order $tord(\gamma, \gamma') \in \mathbb{F} \cup \{\infty\}$ of γ and γ' is the exponent κ in $|\gamma - \gamma'| = ct^\kappa +$ (higher terms), $c \neq 0$.

An arc is **Lipschitz non-singular** if it is an interior arc of a normally embedded Hölder triangle $T \subset X$. There are finitely many Lipschitz singular arcs in $V(X)$.

A Lipschitz singular arc may contain Lipschitz regular points of X.

A Hölder triangle is **non-singular** if all its interior arcs are Lipschitz non-singular.

A **zone** is a set $Z \subset V(X)$ such that for any arcs $\gamma \neq \gamma'$ in Z there is a non-singular Hölder triangle T bounded by γ and γ' such that $V(T) \subset Z$.

The **order** $ord(Z)$ of a zone Z is the infimum of tangency orders of arcs in Z. A **singular** zone $Z = \{\gamma\}$ has order ∞.

A zone Z is **closed** if there are arcs γ and γ' in Z such that $tord(\gamma, \gamma') = ord(Z)$, otherwise Z is **open**.

An arc γ in a β-Hölder triangle $T = T(\gamma_1, \gamma_2)$, bounded by the arcs γ_1 and γ_2, is **generic** if $tord(\gamma, \gamma_1) = tord(\gamma, \gamma_2) = \beta$.

A zone Z is **perfect** if, for any $\gamma \neq \gamma'$ in Z, there is a Hölder triangle T such that $V(T) \subset Z$ and both γ and γ' are generic arcs of T.

Special case: pizza. Let X be the union of a β-Hölder triangle T in the xy-plane and a graph $z = f(x, y)$ of a Lipschitz function f defined on T, such that $f(0, 0) = 0$.

The **order** of f on $\gamma \subset T$ is $ord_\gamma f = tord(\gamma, \gamma')$ where $\gamma' = (\gamma, f(\gamma))$.
Let $Q(T) \subset \mathbb{F} \cup \{\infty\}$ be the set of $q = ord_\gamma f$ for all $\gamma \subset T$.
The set $Q(T)$ is a closed interval in $\mathbb{F} \cup \{\infty\}$.

T is **elementary** if $Z_q = \{\gamma \subset T,\ ord_\gamma f = q\}$ is a zone for any $q \in Q(T)$. The **width function** on $Q(T)$ is defined as $\mu(q) = \mathrm{ord}(Z_q)$.

T is a **pizza slice** if either $Q(T)$ is a single point, or $\mu(q) = aq + b$ is affine, where $a \neq 0$.

The boundary arc $\tilde{\gamma}$ of T where μ is maximal is its **supporting arc**.

A **pizza** on T associated with f is a decomposition of T into Hölder triangles T_j, each of them a pizza slice, with several **toppings:**

- exponent β_j of T_j,
- closed interval $Q_j = Q(T_j)$ in $\mathbb{F} \cup \{\infty\}$,
- affine width function $\mu_j(q)$ on Q_j,
- supporting arc $\tilde{\gamma}_j$ of T_j (when Q_j is not a point),
- sign s_j of f on T_j (not needed if f is non-negative).

A pizza is **minimal** if the union of any two adjacent pizza slices is not a pizza slice.

Theorem (Birbrair *et al.* 17). **The minimal pizza exists and is unique, up to bi-Lipschitz equivalence, for the Lipschitz contact equivalence class of f.**

For a non-negative Lipschitz function f on T, **Lipschitz contact equivalence class** of f is the same as **outer Lipschitz equivalence class** of the union X of T and the graph of f.

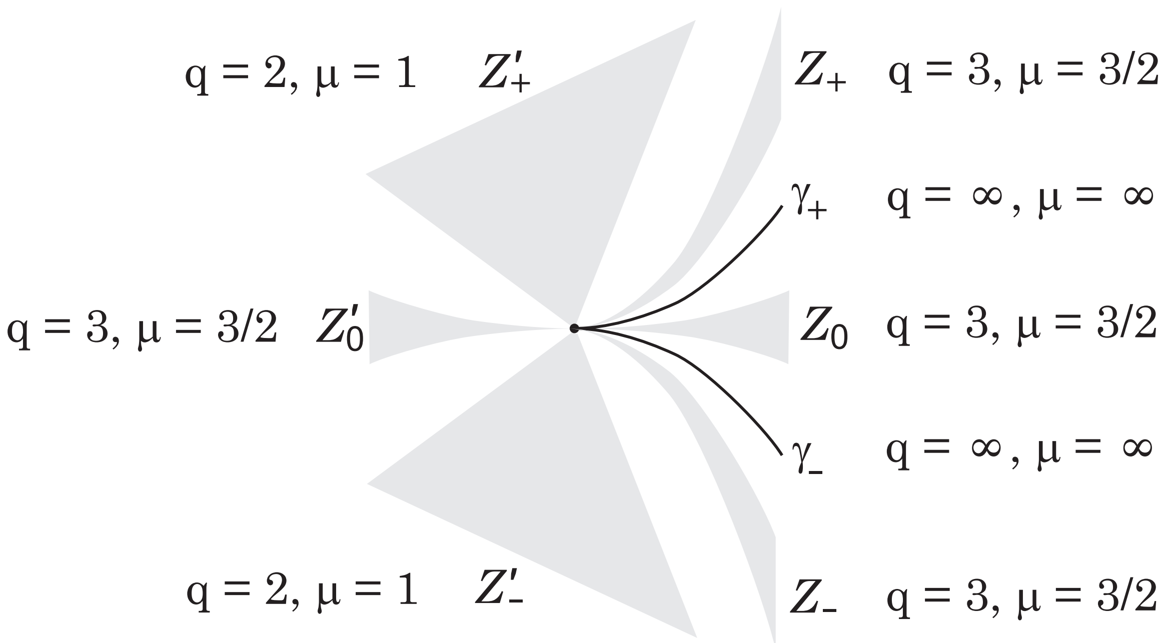
All toppings of a minimal pizza are **canonical**, while the pizza slices T_j are not. However, boundary arcs of Hölder triangles T_j can be placed in **canonical** perfect zones $Z_i \subset V(T)$. **Here is the plan:**

1. Identify a **canonical** finite family of perfect **boundary zones** $Z_i \subset V(T)$ where boundary arcs of T_j can be placed.

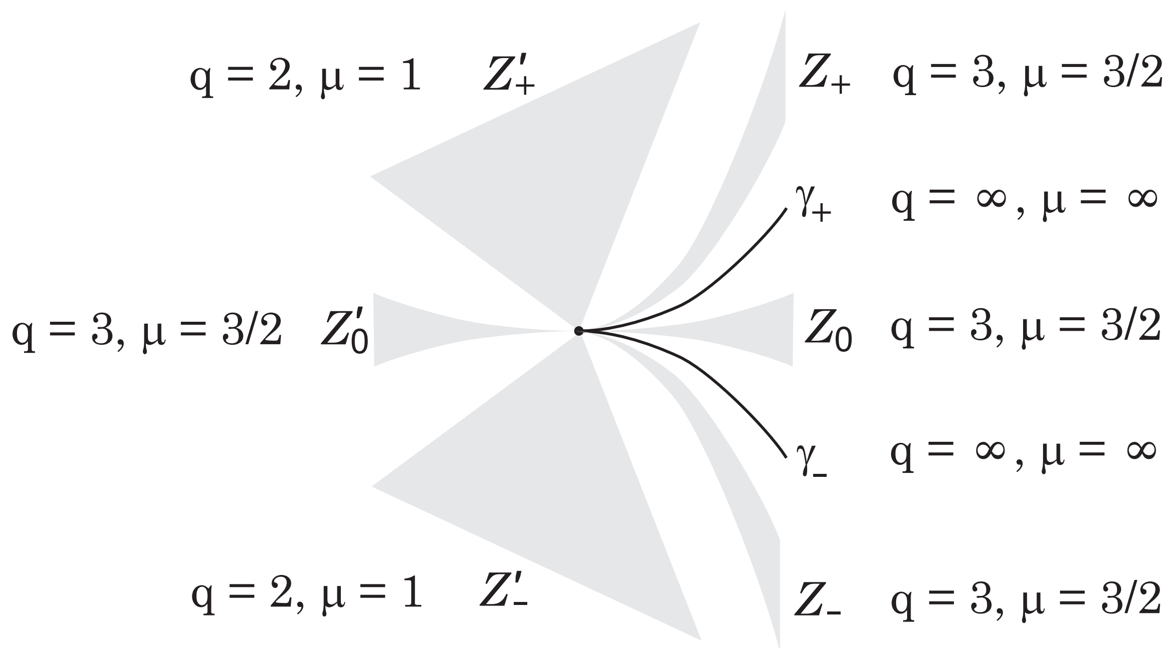
2. Choose **arbitrary boundary arcs** of Hölder triangles T_j inside the zones Z_i. All choices define minimal pizzas for f, resulting in **outer Lipschitz equivalent** decompositions of X.

3. Replace this geometric construction with a canonical **abstract combinatorial object** to get **outer Lipschitz invariant** of X.

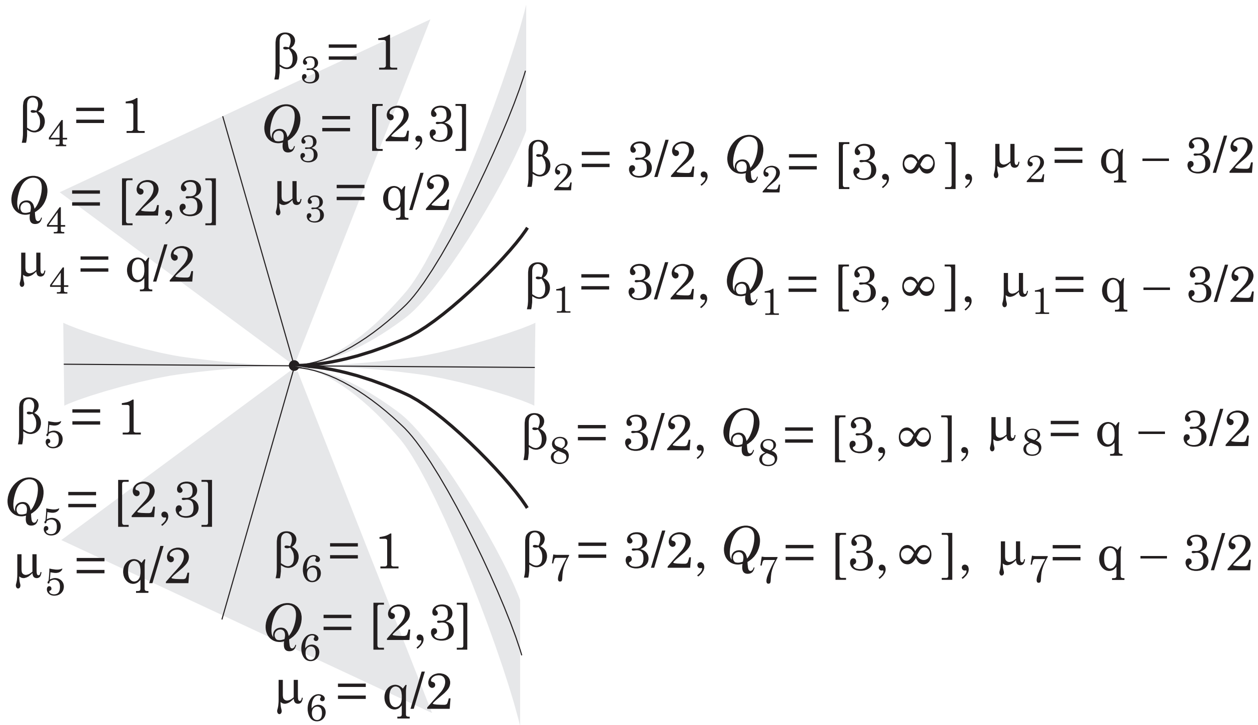
Example: $f(x,y) = y^2 - x^3$. We have $f|_{\gamma_\pm} \equiv 0$, where $\gamma_\pm = \{x \geq 0,\ y = \pm x^{3/2}\}$ are **singular boundary zones**. There are six **boundary zones** of finite order μ:



The set of arcs γ such that $q = \text{ord}_\gamma f = 3$ consists of four zones: Z_+, Z_-, Z_0, Z'_0. Each of them is perfect of order $\mu = 3/2$.
The set of arcs γ such that $q = \text{ord}_\gamma f = 2$ consists of two zones: Z'_+ and Z'_-. Each of them is perfect of order $\mu = 1$.



A minimal pizza for f consists of eight slices T_j with the boundary arcs γ_+, γ_- and an arbitrary arc selected in each of the six other boundary zones.



General surface germ X: normal and abnormal zones.

A Lipschitz non-singular arc $\gamma \subset X$ is **abnormal** if there are normally embedded Hölder triangles T and T' in X such that $\gamma = T \cap T'$ and $T \cup T'$ is not normally embedded. Otherwise, γ is **normal**.

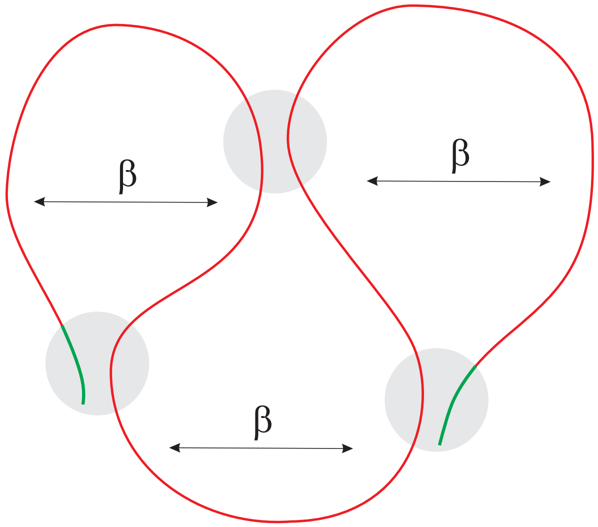
A zone $Z \subset V(X)$ is **abnormal** (resp., **normal**) if all arcs in Z are abnormal (resp., normal).

An abnormal (resp., normal) zone is **maximal** if it is not contained in a larger abnormal (resp., normal) zone.

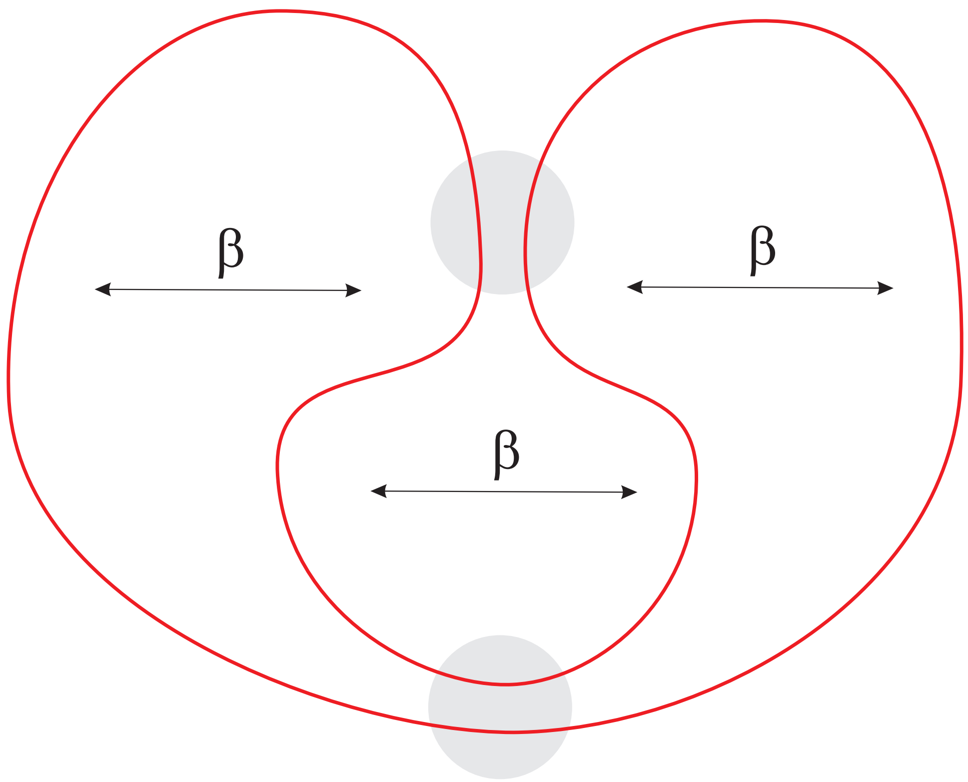
Theorem (AG, Souza 21) **For any surface germ X, there is a canonical partition of $V(X)$ into finitely many maximal abnormal and maximal normal zones. All normal zones are normally embedded. All maximal abnormal zones are closed perfect and weakly normally embedded.**

Snakes, circular snakes, bubble snakes and non-snake bubbles.

A β-**snake** is a non-singular β-Hölder triangle T such that each arc in $V(T)$ is abnormal iff it is generic. A maximal abnormal β-zone $Z \subset V(X)$ is a **snake zone** if there is a β-Hölder triangle $T \subset X$ such that $Z \subset V(T)$.



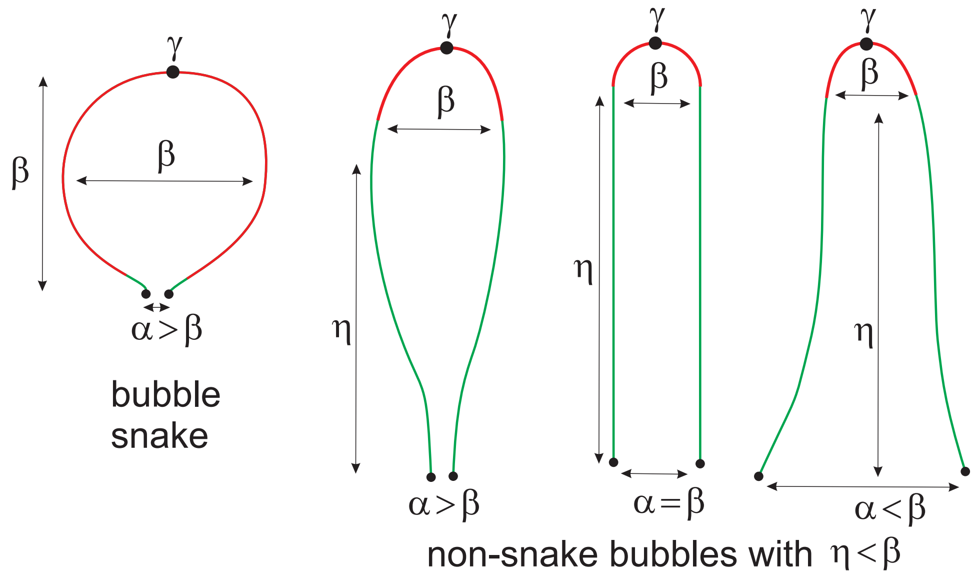
A **circular** β**-snake** is a β-horn C such that all arcs in the **circular** β**-snake zone** $V(C)$ are abnormal.



A **bubble** is a Hölder triangle T bounded by γ_1 and γ_2, such that $tord(\gamma_1, \gamma_2) > itord(\gamma_1, \gamma_2)$, partitioned into normally embedded triangles by an arc γ. A **bubble snake** is a bubble that is a snake.

A **non-snake bubble** is a bubble that does not contain a snake.

A **non-snake abnormal zone** is a maximal abnormal zone $Z \subset V(T)$ where T is a non-snake bubble.



Theorem (AG, Souza 21) **Each maximal abnormal β-zone in $V(X)$ is either a β-snake zone, or $V(C)$ where C is a circular β-snake, or a non-snake abnormal β-zone $Z \subset V(T)$ where T is a non-snake η-bubble for some $\eta < \beta$.**

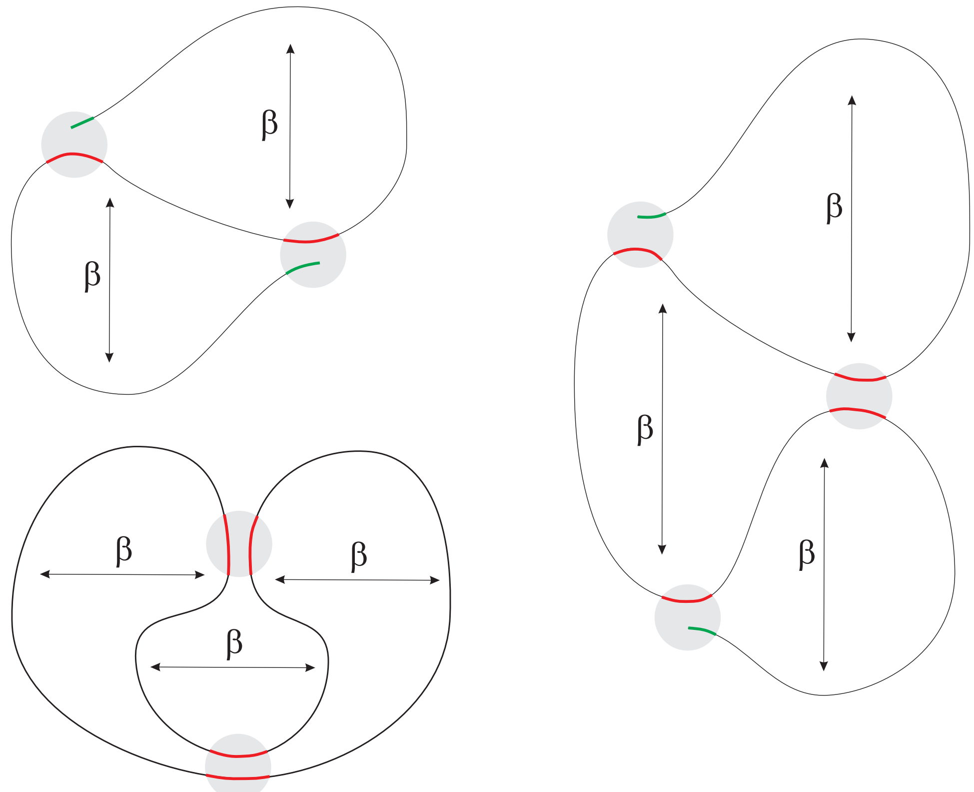
Each non-snake abnormal β-zone is closed perfect and normally embedded.

Each maximal β-snake zone, and each circular β-snake zone, has a canonical partition into finitely many normally embedded β-zones: closed perfect **segments** and open perfect **nodal zones**.

A **boundary nodal zone** of a β-snake zone Z is a normal zone where a boundary arc of a β-snake T may be placed, so that Z is the set of generic arcs of T.

Each boundary nodal zone is an open, normally embedded β-zone. If it is not perfect, then it contains either a Lipschitz singular arc or a perfect boundary nodal zone of an α-snake, for some $\alpha > \beta$, or it is adjacent to a non-snake abnormal α-zone, for some $\alpha > \beta$.

Segments, nodal zones (red) and boundary nodal zones (green).



Outer Lipschitz invariant decomposition of a surface germ X.

A pair (T, T') of normally embedded Hölder triangles is **transversal** if $T \cup T'$ is a subset of a normally embedded triangle.

A non-transversal pair (T, T') is **coherent** if it is outer Lipschitz equivalent to the union of a pizza slice T for a Lipschitz function f and the graph T' of f over T.

Step 1. Define canonical **primary zones** in $V(X)$ as

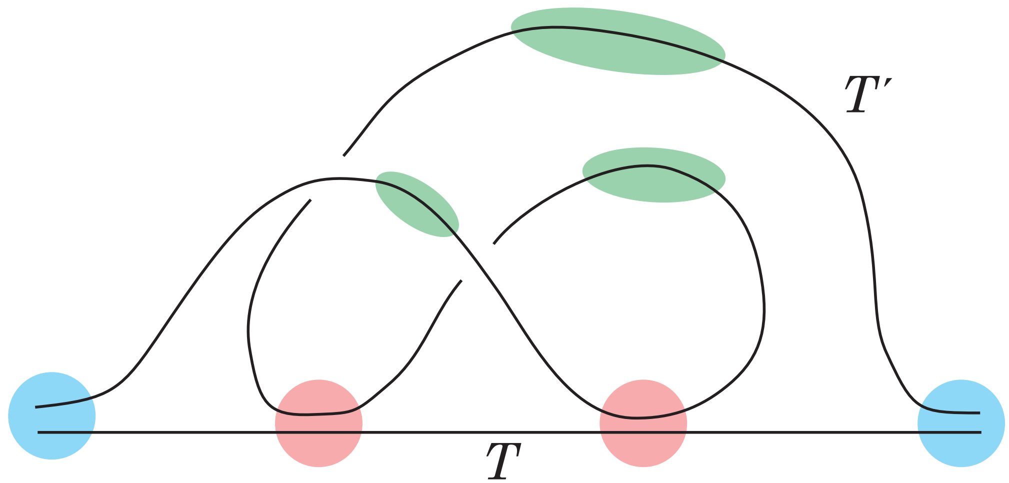
- Lipschitz singular arcs,
- Maximal normal zones,
- Segments and nodal zones of snake zones and circular snakes,
- Boundary nodal zones of snakes,
- Non-snake abnormal zones.

Step 2. Using pizza decomposition for the “distance functions” between primary zones, define **secondary zones** in $V(X)$ so that minimal (by inclusion) secondary zones are perfect.

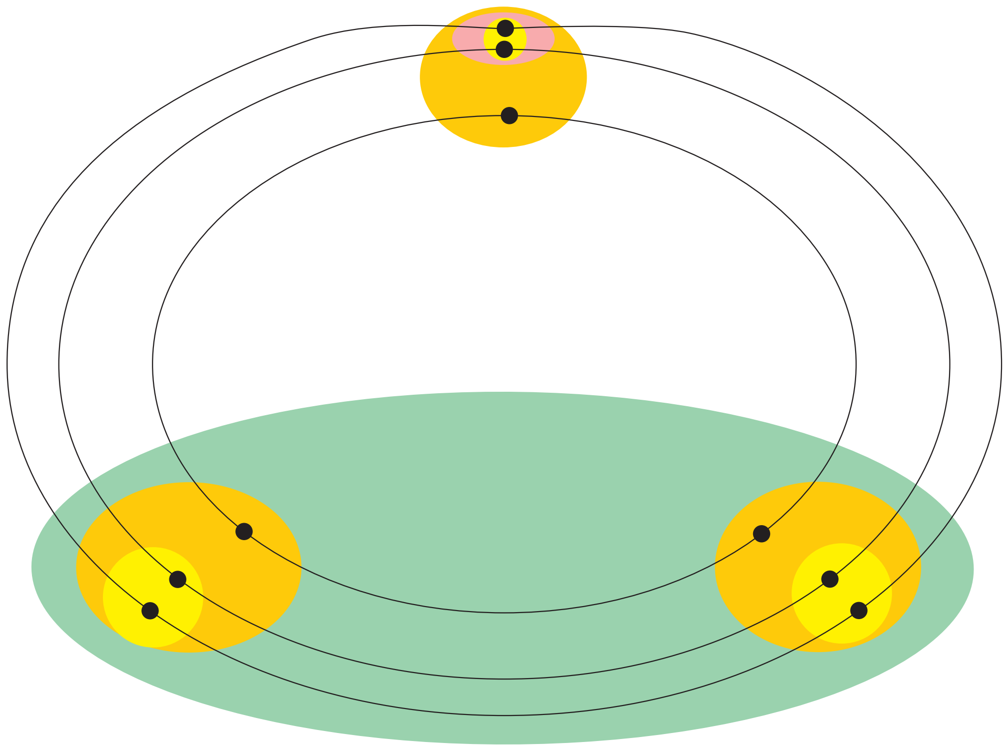
Step 3. Placing **boundary arcs** in minimal secondary zones, decompose X into finitely many isolated arcs and normally embedded Hölder triangles, so that any two triangles are either coherent or transversal, and all choices of arcs result in outer Lipschitz equivalent decompositions.

This step involves **proper order** in which the boundary arcs are placed, starting with isolated arcs and Lipschitz singular arcs.

Extra **tertiary zones** are associated with some boundary arcs, where more boundary arcs should be placed.



Example: Secondary zones in Step 2



Example: Tertiary zones in Step 3

Main Theorem.

There is a unique up to outer Lipschitz equivalence decomposition of a surface germ X into isolated arcs and normally embedded Hölder triangles, such that any two triangles are either coherent or transversal.

Two such decompositions are combinatorially equivalent if there is one-to-one correspondence between their arcs and triangles, preserving adjacency relations, tangency exponents between any two isolated arcs and/or boundary arcs of triangles, and pizza toppings for the distance between any two coherent triangles: interval $Q \subset \mathbb{F} \cup \{\infty\}$, width function $\mu(q) = aq + b$ on Q, supporting arc $\tilde{\gamma}$.

Two surface germs are outer Lipschitz equivalent if and only if their canonical decompositions are combinatorially equivalent.