Lipschitz Geometry of Pairs of Normally Embedded Hölder Triangles

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All sets and maps are **definable** in a polynomially bounded o-minimal structure over $\mathbb{R}$ with the field of exponents $\mathbb{F}$, e.g., **semialgebraic** or **subanalytic** with $\mathbb{F} = \mathbb{Q}$.

A set $X \subset \mathbb{R}^n$ inherits from $\mathbb{R}^n$ two metrics:
the **outer metric** $dist(x, y) = |y - x|$ and the **inner metric** $idist(x, y) =$ length of the shortest path in X connecting x and y.

The set X is **Lipschitz Normally Embedded (LNE)** if these two metrics on X are equivalent.

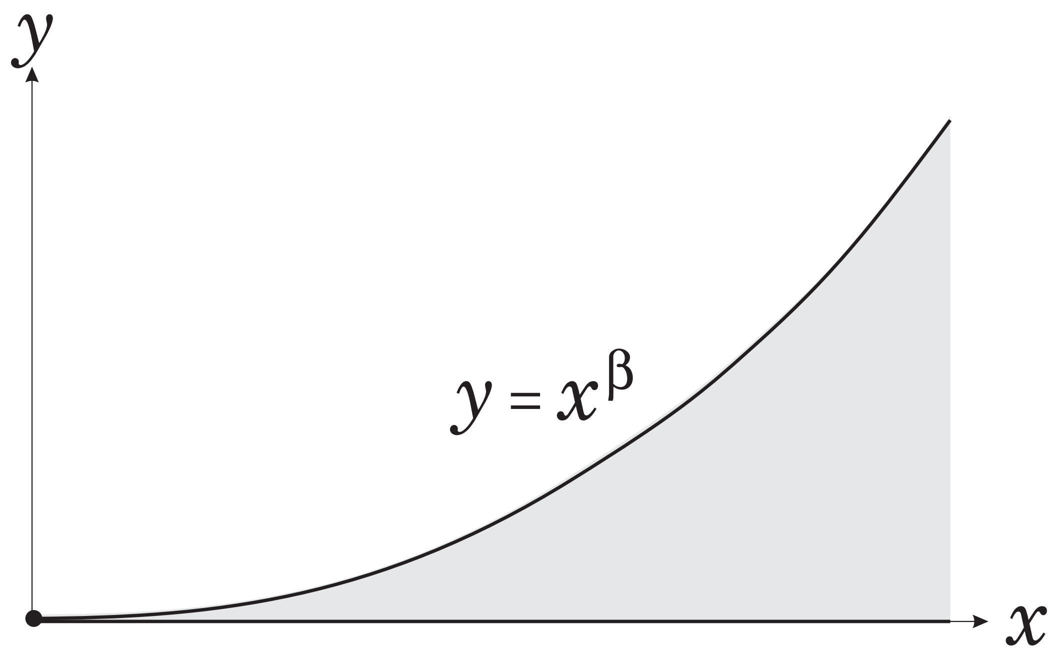
A **surface germ** X is a closed two-dimensional germ at the origin.

Two surface germs X and Y are outer (inner) **Lipschitz equivalent** if there is an outer (inner) bi-Lipschitz homeomorphism $h : X \to Y$.

A **standard** β-**Hölder triangle**, for $1 \leq \beta \in \mathbb{F}$, is the set

$$T_\beta = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ 0 \leq y \leq x^\beta\}.$$

The curves $\{y = 0\}$ and $\{y = x^\beta\}$ are the **boundary arcs** of T_β.



A β-**Hölder triangle** $T = T(\gamma_1, \gamma_2)$ with **boundary arcs** γ_1 and γ_2 is a surface germ inner Lipschitz equivalent to T_β.

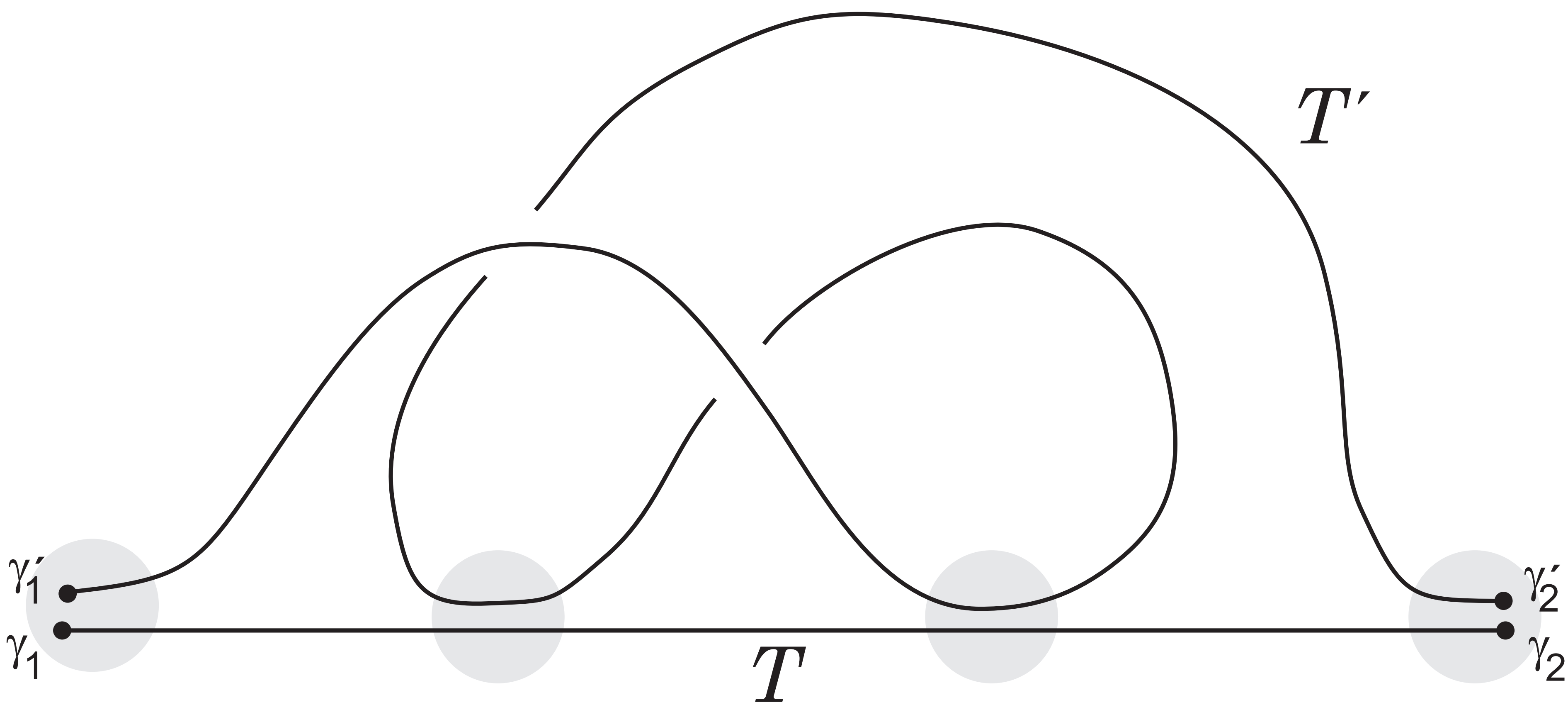
A single LNE β-Hölder triangle is the simplest surface germ: given $1 \leq \beta \in \mathbb{F}$, all LNE β-Hölder triangles are outer Lipschitz equivalent.

We consider the next simplest case: a **pair** (T, T') of LNE β-Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$.

Trivial Example: A **special** pair (T, T') where T is LNE β-Hölder triangle and T' is a graph of a Lipschitz function defined on T.

The outer Lipschitz geometry of pairs of LNE Hölder triangles is surprisingly non-trivial.

Our goal: Outer Lipschitz **classification** of pairs of LNE Hölder triangles, based on a **discrete invariant** of such pairs, such that two pairs of LNE Hölder triangles are outer Lipschitz equivalent if, and only if, they have the same invariant.



Non-trivial Example: The link of a pair of LNE β-Hölder triangles. Shaded disks indicate zones with the tangency order higher than β.

Valette link

An **arc** γ in a surface germ X is a germ of a map $\gamma : [0, \epsilon) \to X$ such that $|\gamma(t)| = t$. We write $\gamma \subset X$ identifying γ with its image in X. The **Valette link** $V(X)$ is the space of all arcs in X.

The **tangency order** $tord(\gamma, \gamma')$ of two arcs γ and γ' is the exponent $\kappa \in \mathbb{F} \cup \{\infty\}$ in $|\gamma(t) - \gamma'(t)| = ct^\kappa +$ (higher terms), where $c \neq 0$. We define $tord(\gamma, Y) = \sup_{\lambda \in V(Y)} tord(\gamma, \lambda)$ for a surface germ Y.

The tangency order defines a **non-archimedean metric** on $V(X)$.

An arc $\gamma \subset X$ is **Lipschitz non-singular** if it is an **interior** arc (not a boundary arc) of a LNE Hölder triangle $T \subset X$.
There are finitely many Lipschitz singular arcs in $V(X)$.

A Hölder triangle is **non-singular** if all its interior arcs are Lipschitz non-singular.

Zones

For a surface germ X, a non-empty subset $Z \subset V(X)$ is a **zone** if, for any two arcs $\gamma \neq \gamma'$ in Z, there is a non-singular Hölder triangle $T = T(\gamma, \gamma')$ such that $V(T) \subset Z$.

The **order** $\mu(Z)$ of a zone Z is the infimum of tangency orders of arcs in Z. A **singular** zone $Z = \{\gamma\}$ has order ∞.

A zone Z is **closed** if there are arcs γ and γ' in Z such that $tord(\gamma, \gamma') = \mu(Z)$ and **open** otherwise.

If $T = T(\gamma_1, \gamma_2)$ is a β-Hölder triangle, then an arc $\gamma \subset T$ is **generic** if $tord(\gamma, \gamma_1) = tord(\gamma, \gamma_2) = \beta$. A zone $Z \subset V(X)$ is **perfect** if, for any arcs $\gamma \neq \gamma'$ in Z, there is a Hölder triangle $T \subset X$ such that $V(T) \subset Z$ and both γ and γ' are generic arcs of T.

Pizza

Let $T \subset \mathbb{R}^2_{xy}$ be a β-Hölder triangle, $f(x, y)$ a non-negative Lipschitz function on T such that $f(0,0) = 0$, and let $T' = \{z = f(x, y)\}$ be the graph of f.

For an arc $\gamma \subset T$, let $\gamma' \subset T'$ be the graph of $f|_\gamma$. The **order** of f on γ is the exponent $ord_\gamma f = tord(\gamma, \gamma')$.

Let $Q(T) \subset \mathbb{F} \cup \{\infty\}$ be the set of exponents $q = ord_\gamma f$ for all $\gamma \subset T$. The set $Q(T)$ is a closed interval in $\mathbb{F} \cup \{\infty\}$.

T is **elementary** if $Z_q = \{\gamma \subset T,\ ord_\gamma f = q\}$ is a zone for any $q \in Q(T)$. The **width function** on $Q(T)$ is defined as $\mu(q) = \mu(Z_q)$.

T is a **pizza slice** if either $Q(T)$ is a point, or $\mu(q) = aq + b$ is affine. In the latter case $a \neq 0$ and the **supporting arc** $\tilde{\gamma}$ of T is one of its boundary arcs where μ is maximal.

A **pizza** on T associated with a non-negative Lipschitz function f is a decomposition of T into pizza slices T_j with several **toppings:**

• exponent β_j of T_j,

• closed interval $Q_j = Q(T_j)$ in $\mathbb{F} \cup \{\infty\}$,

• affine width function $\mu_j(q) \not\equiv const$ on Q_j such that $\mu_j(q) \le q$ and $\min_{q \in Q_j} \mu_j(q) = \beta_j$ (a single exponent $\mu_j = \beta_j$ if Q_j is a point),

• supporting arc $\tilde{\gamma}_j$ of T_j (when Q_j is not a point).

A pizza is **minimal** if the union of any two adjacent pizza slices is not a pizza slice.

Theorem (Birbrair *et al.*, 2017). **The minimal pizza on a Hölder triangle T associated with a non-negative Lipschitz function f exists, and is unique, up to bi-Lipschitz equivalence, for the Lipschitz contact equivalence class of f.**

Remark: For a non-negative Lipschitz function f on a β-Hölder triangle $T \subset \mathbb{R}^2$, the Lipschitz contact equivalence class of f is the same as the outer Lipschitz equivalence class of the **special** pair (T, T') of LNE β-Hölder triangles, where $T' = \{z = f(x, y)\}$.

All toppings of a minimal pizza are **canonical** (outer Lipschitz invariant) while the pizza slices T_j are not. However, the boundary arcs of the Hölder triangles T_j belong to **canonical perfect pizza zones** in $V(T)$. Two of these zones are the boundary arcs of T.

Choosing any arc in each interior pizza zone as a common boundary arc of two adjacent pizza slices, one gets a minimal pizza for f. Different choices define equivalent pizza decompositions of T.

The toppings of a minimal pizza for a function f on T define a **discrete invariant** of the outer Lipschitz geometry of a special pair (T, T') of LNE Hölder triangles.

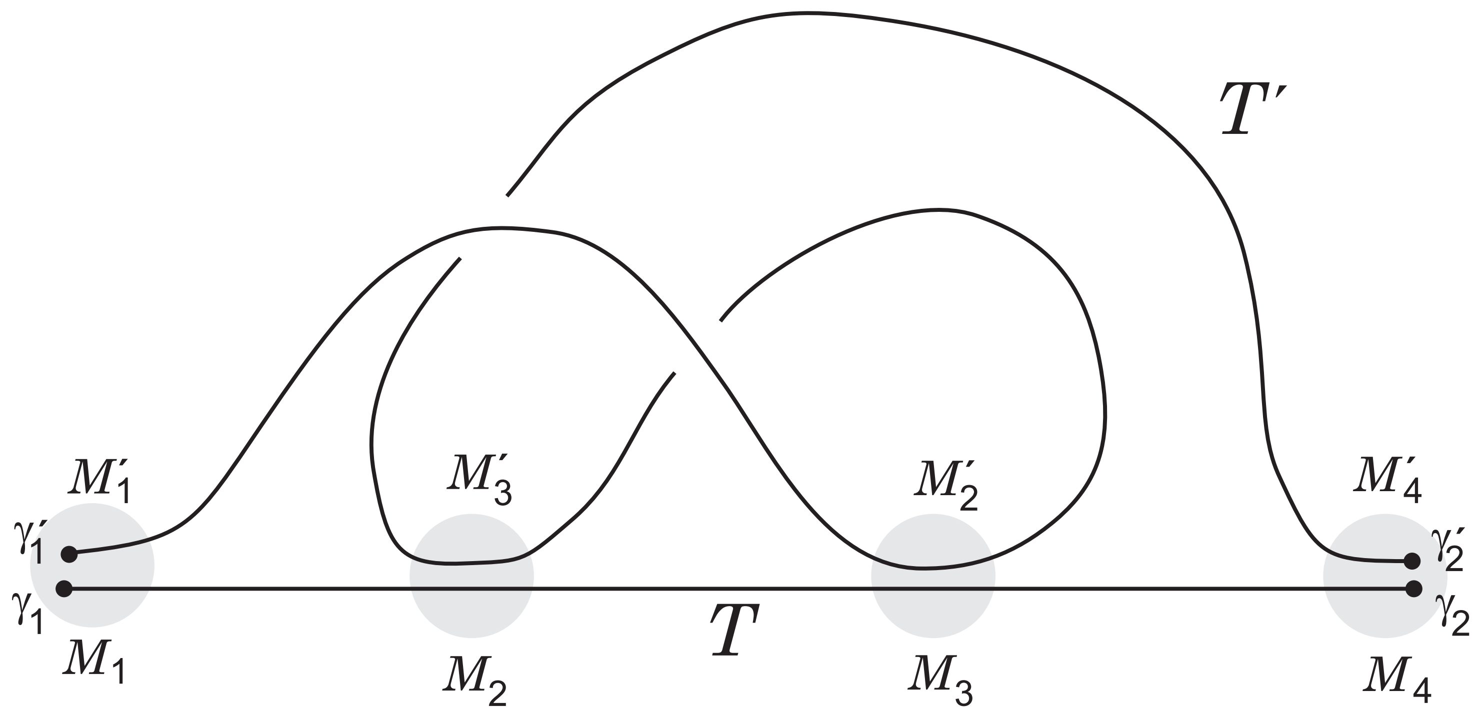
Normal pairs of LNE Hölder Triangles

Given two Hölder triangles T and T', a pair of arcs $\gamma \subset T$ and $\gamma' \subset T'$, is **normal** if $tord(\gamma, T') = tord(\gamma, \gamma') = tord(\gamma', T)$. A pair (T, T') of LNE Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ is **normal** if both pairs (γ_1, γ'_1) and (γ_2, γ'_2) of their boundary arcs are normal.

For example, if T' is a graph of a Lipschitz function f on T, then any pair of arcs (γ, γ'), where $\gamma \subset T$ and $\gamma' \subset T'$ is the graph of $f|_\gamma$, is normal, and the special pair (T, T') of Hölder triangles is normal.

Theorem (Birbrair, AG). **Let (T, T') be a normal pair of LNE Hölder triangles, such that T is elementary with respect to $f(x) = dist(x, T')$. Then (T, T') is outer Lipschitz equivalent to a special pair (T, Γ), where Γ is the graph of f.**
Moreover, T' is elementary with respect to $g(x') = dist(x', T)$, and a minimal pizza for g on T' is equivalent to a minimal pizza for f on T.

If (T, T') is a normal pair of LNE Hölder triangles such that T is not elementary with respect to $f(x) = dist(x, T')$, then $T \cup T'$ may be not equivalent to the union of T and a graph of a function on T.



Example: Permutation of maximum zones of a normal pair (T, T').

Maximum zones

Let (T, T') be a normal pair of LNE Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$. Let $\{D_\ell\}_{\ell=0}^{p}$ be the pizza zones in $V(T)$ of a minimal pizza on T associated with $f(x) = dist(x, T')$, ordered from $D_0 = \{\gamma_1\}$ to $D_p = \{\gamma_2\}$. The **exponent** $q_\ell = tord(D_\ell, T')$ of the zone D_ℓ is defined as $ord_\gamma f$ for $\gamma \in D_\ell$ (it is the same for all $\gamma \in D_\ell$).

A zone D_ℓ is a **maximum zone** if either $0 < \ell < p$ and $q_\ell \geq \max(q_{\ell-1}, q_{\ell+1})$, or $\ell = 0$ and $q_0 \geq q_1$, or $\ell = p$ and $q_p \geq q_{p-1}$.

Maximum zones in $V(T')$ are some of the pizza zones D'_ℓ of a minimal pizza on T' associated with $g(x') = dist(x', T)$. They are defined similarly, exchanging T and T'.

Theorem (Birbrair, AG). **Let (T,T') be a normal pair of LNE Hölder triangles $T=T(\gamma_1,\gamma_2)$ and $T'=T(\gamma'_1,\gamma'_2)$ oriented from γ_1 to γ_2 and from γ'_1 to γ'_2, respectively.**

Let $\{M_i\}_{i=1}^m$ and $\{M'_j\}_{j=1}^n$ be the maximum zones in $V(T)$ and $V(T')$, ordered according to the orientations of T and T'. Let $\bar{q}_i = tord(M_i,T')$ and $\bar{q}'_j = tord(M'_j,T)$.

Then $m=n$, and there is a canonical permutation

$$\sigma : [1,\dots,m] \to [1,\dots,m]$$

such that $ord(M_i) = ord(M'_{\sigma(i)})$ and $tord(M_i, M'_{\sigma(i)}) = \bar{q}_i = \bar{q}'_{\sigma(i)}$.

If $\{\gamma_1\} = M_1$ is a maximum zone, then $\{\gamma'_1\} = M'_1$ and $\sigma(1)=1$.
If $\{\gamma_2\} = M_m$ is a maximum zone, then $\{\gamma'_2\} = M'_m$ and $\sigma(m)=m$.

Transversal Hölder triangles and coherent pizza slices

Two LNE Hölder triangles T and T' are **transversal** if there is a boundary arc $\tilde{\gamma}$ of T and a boundary arc $\tilde{\gamma}'$ of T' such that $tord(\gamma',\tilde{\gamma}) = tord(\gamma',T)$ for all arcs γ' of T' and $tord(\gamma,\tilde{\gamma}') = tord(\gamma,T')$ for all arcs γ of T.

A pizza slice T_j of a pizza decomposition of a Hölder triangle T associated with a function f is **transversal** if T_j and the graph of $f|_{T_j}$ are transversal Hölder triangles.

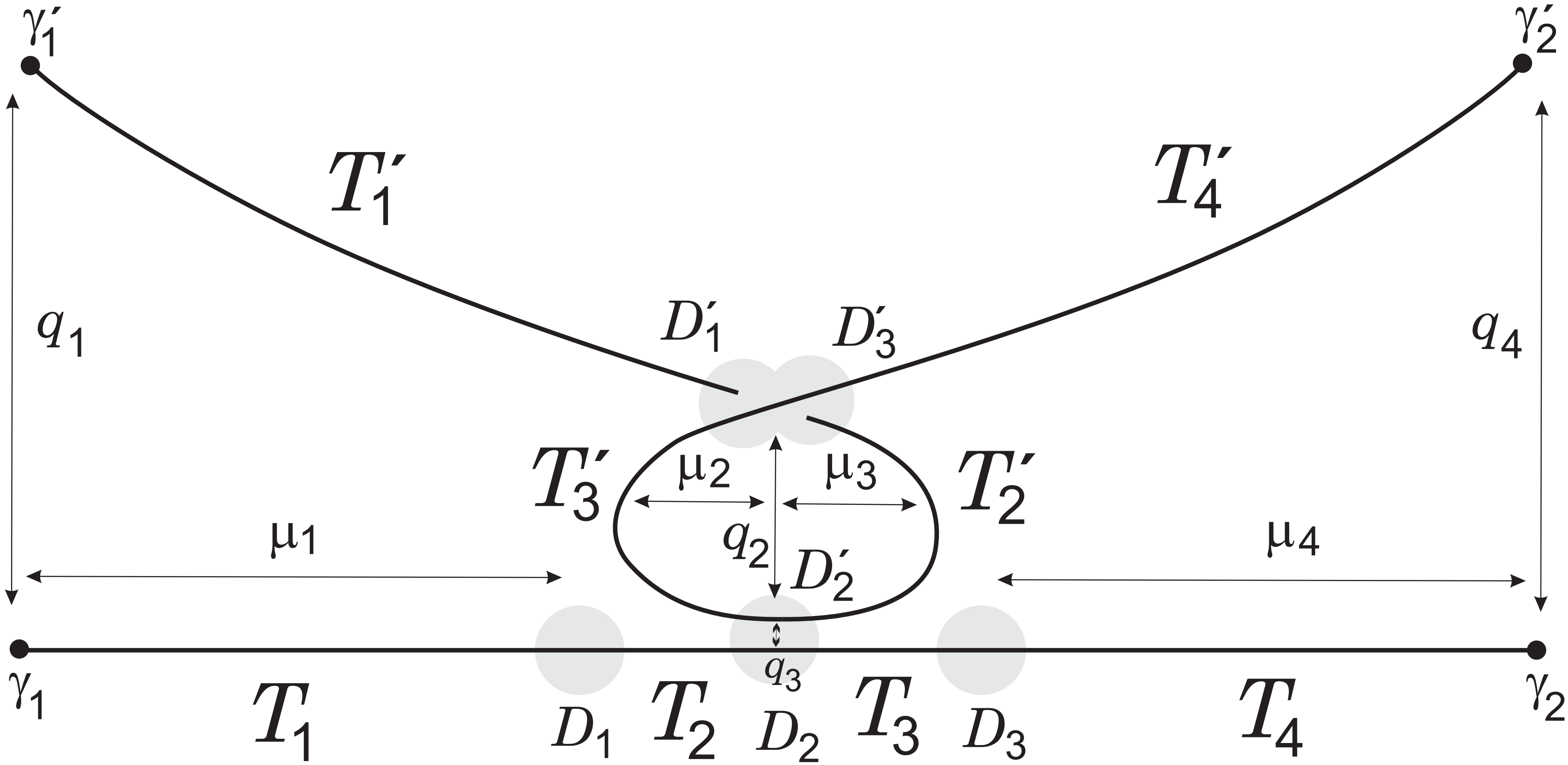
Alternatively, a pizza slice T_j is transversal if $\mu_j(q) \equiv q$, where μ_j is the affine width function on Q_j associated with f.

A pizza slice T_j is **coherent** if it is not transversal, thus $\mu_j(q) \not\equiv q$.

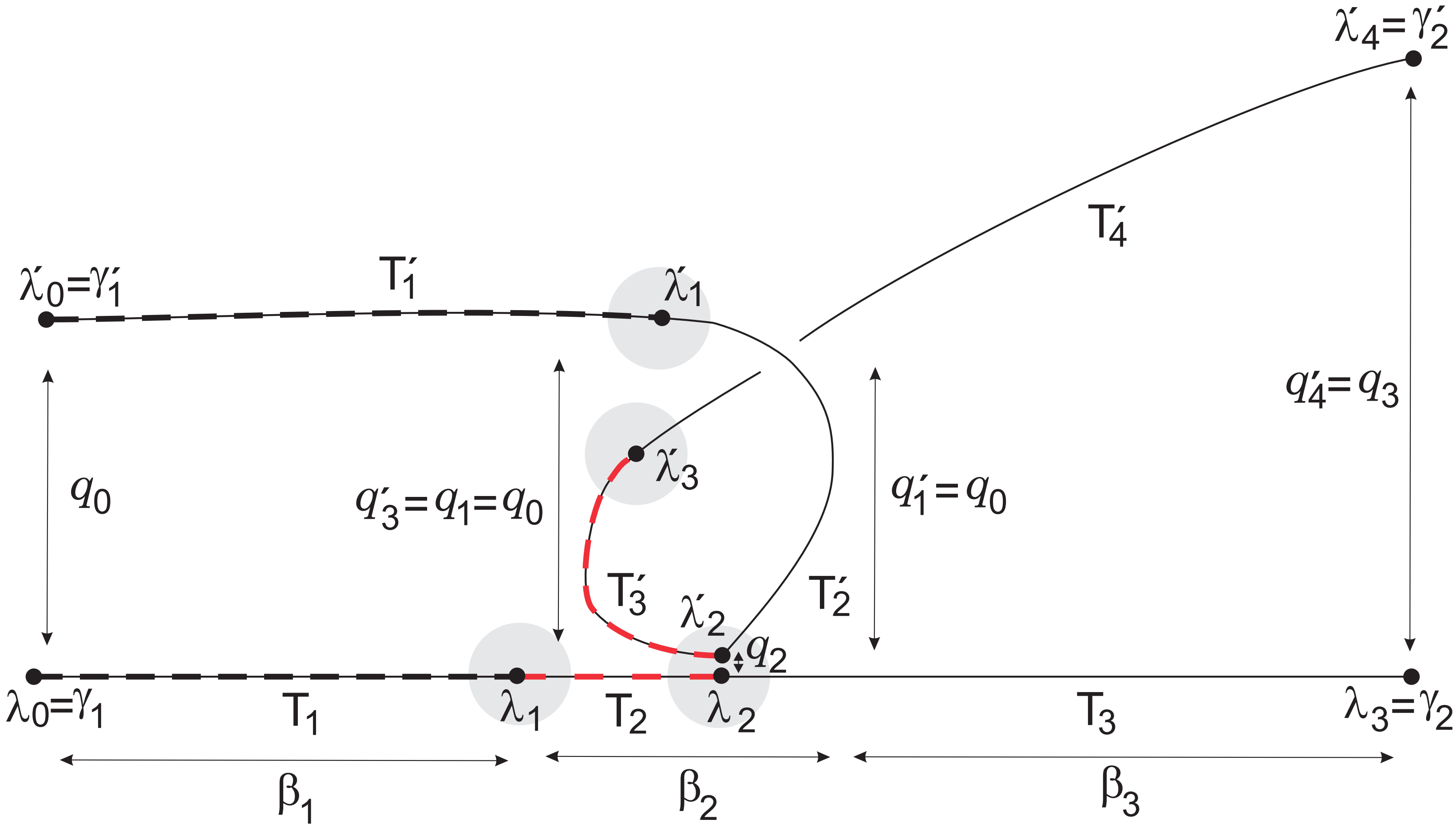
Theorem (Birbrair, AG). **Let (T, T') be a normal pair of LNE Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$, oriented from γ_1 to γ_2 and from γ'_1 to γ'_2, respectively. Let $\{T_i\}_{i=1}^p$ and $\{T'_j\}_{j=1}^s$ be minimal pizza decompositions of T and T' associated with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$.**

Then there is a canonical one-to-one correspondence $j = \tau(i)$ between coherent pizza slices T_i of T and coherent pizza slices T'_j of T', such that each pair (T_i, T'_j), where $j = \tau(i)$, is outer Lipschitz equivalent to the pairs (T_i, Γ) and (T'_j, Γ'), where Γ is the graph of $f|_{T_i}$ and Γ' is the graph of $g|_{T'_j}$.

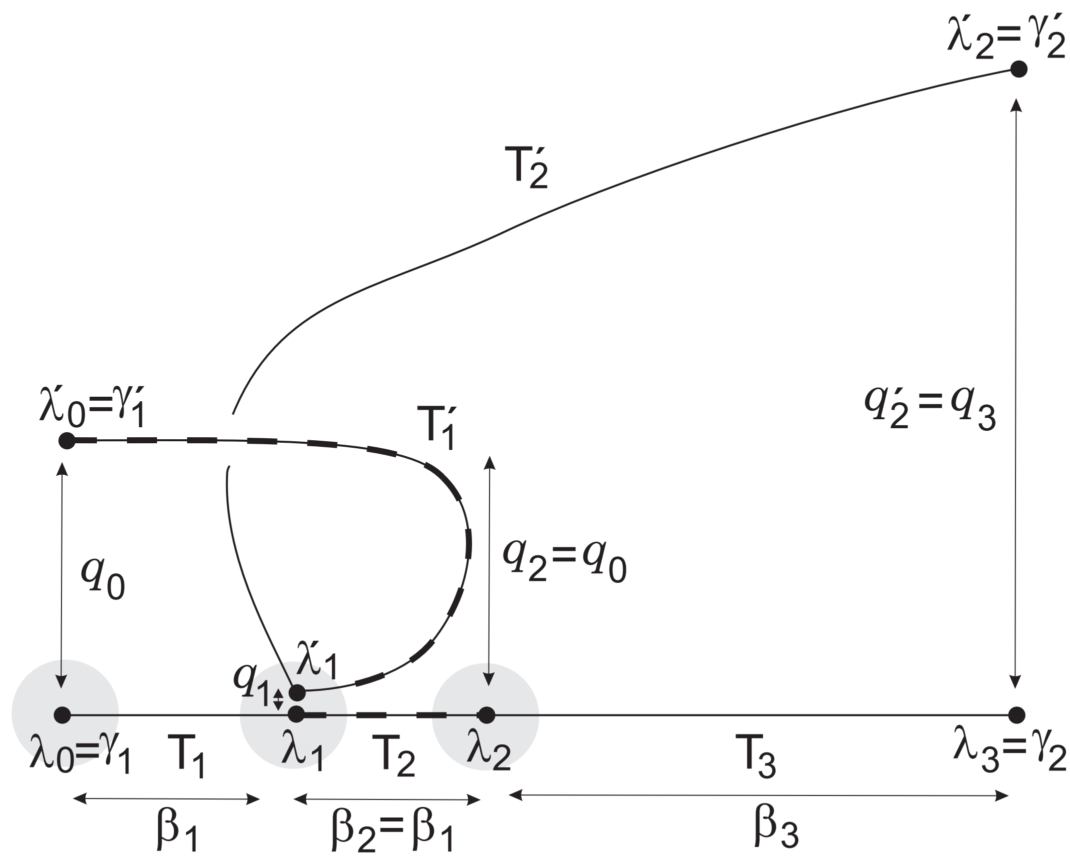
The pair (T_i, T'_j), where $j = \tau(i)$, is **positive** if orientations of T_i and T'_j induced by the correspondence τ are consistent with orientations of T and T'. Otherwise, the pair (T_i, T'_j) is **negative**.
Thus τ is a **signed** correspondence.



Example: A normal pair of LNE Hölder triangles with two positive pairs (T_1, T_1') and (T_4, T_4') of coherent pizza slices, and two negative pairs (T_2, T_3') and (T_3, T_2').



Example: A normal pair of Hölder triangles with different number of pizza slices in the minimal pizzas on T and T' associated with the distance functions $f = dist(x, T')$ and $g = dist(x', T)$.



Example: A normal pair of Hölder triangles with different number of pizza slices in the minimal pizzas on T and T' associated with the distance functions $f = dist(x, T')$ and $g = dist(x', T)$.

Theorem (Birbrair, AG). **Two normal pairs (T, T') and (S, S') of LNE Hölder triangles are outer Lipschitz equivalent if, and only if, the following holds:**

1. Minimal pizzas on T and T' associated with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$ are equivalent to minimal pizzas on S and S' associated with the distance functions $\phi(s) = dist(s, S')$ and $\psi(s') = dist(s', S)$, respectively.

2. The numbers of maximum zones for the pairs (T, T') and (S, S') are equal, and the permutation σ of the maximum zones for the pair (S, S') is the same as for the pair (T, T').

3. The numbers of coherent pizza slices for the pairs (T, T') and (S, S') are equal, and the signed correspondence τ between coherent pizza slices for the pair (S, S') is the same as for the pair (T, T').