Zero-one laws of finitely presented structures

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Let φ be a first-order sentence in the language of graphs. Then $Pr(G(n,p) \vDash \varphi) \to 0$ or 1 as $n \to \infty$. Furthermore, the probability is 1 iff the random graph $G(\infty, p) \vDash \varphi$.

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Conjecture (Knight, '13)

A first-order sentence is true in a free group iff it is true in a random group.

A toy example

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Theorem

For every sentence φ, φ is true in a 1-generated random structure iff it is true in the 1-generated free structure.

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Example

Groups and rings are algebraic varieties.

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- The variety with a pair of inverse functions satisfies the strong zero-one law.

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In the variety with a pair of inverse functions:

1. Random identities cannot be detected locally
2. Every sentence is equivalent to a Boolean combination of local sentences

Another example

Example

The variety with $L = \{f(x), g(x)\}$ and $T = \varnothing$ does not satisfy the 0-1 law.

Yet another example

Example

The variety with $L = \{f(x)\}$ and $T = \varnothing$ satisfy the 0-1 law, but the limiting theory differs from the theory of the free structure.

Gaifman's Locality Theorem

Definition

Let A be a relational structure. The *Gaifman graph* of A is the graph with $V = A$ and $(a, b) \in E$ if there is some R with $R(\overline{x})$ and $a, b \in \overline{x}$.

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Theorem (Gaifman Locality Theorem, '82)

Let L be a relational language. Then every sentence is equivalent to a Boolean combination of sentences of the form

$$\exists v_1, \cdots, v_s \left(\bigwedge_i \alpha_i^{(r)}(v_i) \wedge \bigwedge_{i<j} d(v_i, v_j) > 2r \right).$$

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$d(x, y)$: the distance function of the graph

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If $T \supseteq$ "f_i, f_j commute" and "f_i^{-1} is the inverse of f_i", then the variety satisfies the strong 0-1 law.

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Question

What if we drop commutivity?

non-bijective structures

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If there are two elements x_1 and x_2 in the free structure such that a random term equals x_i with a positive probability, then the variety does not satisfy the 0-1 law.

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Example

Let $T = \{\forall x\ f^n(x) = x\}$.
A random structure in this variety is trivial with probability $\phi(n)/n$.

Further questions

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The analogue of Tarski's problem in varieties:
When are the free structures in a variety elementarily equivalent?
When are the standard embeddings elementary?