METRIC KNOT THEORY AND AMBIENT LIPSCHITZ GEOMETRY OF SURFACE GERMS

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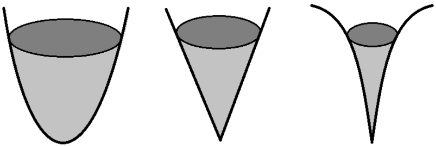
Lipschitz Classification Problem

Lipschtz Geometry of Singularities is an intensively developing part of Singularity Theory. The problem of classification of singular sets up to bi-Lipschitz equivalences are closely related to topological, differential and analytical equivalence of sets.



Let us first know the preliminary definitions in Lipschitz Geometry.

Lipschitz Metrics

Given $C \geq 1$, metric spaces (X, d_1) e (Y, d_2), a map $\varphi : X \to Y$ is **bi-Lipschitz** (or C-bi-Lipschitz) if φ is bijective and if, for all $p, q \in X$:

$$\frac{1}{C} \cdot d_1(p,q) \leq d_2(\varphi(p), \varphi(q)) \leq C \cdot d_1(p,q)$$

Given an connected semialgebraic set $X \subseteq \mathbb{R}^n$, there are two natural metrics on it:

1. **Outer Metric**: $d(p,q) = \|p - q\|, \forall\, p, q \in X$ (euclidian distance)
2. **Inner Metric**: $d_X(p,q) = \inf\{l(p,q)\}, \forall\, p, q \in X$, where the infimum is considered over all rectifiable paths connecting p to q.

Bi-Lipschitz Equivalences

Bi-Lipschitz Equivalence: We say that $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ are:

1. **Outer Bi-Lipschitz Equivalent**: There is $\varphi : X \to Y$ bi-Lipschitz on the outer metric;
2. **Inner Bi-Lipschitz Equivalent**: There is $\varphi : X \to Y$ bi-Lipschitz on the inner metric;
3. **Ambient Bi-Lipschitz Equivalent**: There is $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ outer bi-Lipschitz, such that $\varphi(X) = Y$ (clearly $m = n$).

Normally Embedded Sets

We say that a set $X \subseteq \mathbb{R}^n$ is **Lipschitz Normally Embedded (LNE)** if the outer metric and the inner metric are equivalent. Analogous definitions can be done to germs of sets in a given point.

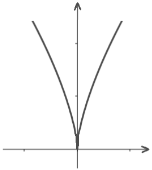
Arcs: An arc in $\mathbb{R}^n$ is a germ at the origin of a semialgebraic map $\gamma : [0, t_0) \to \mathbb{R}^n$, for some $t_0 > 0$ sufficiently small, such that $\gamma(0) = 0$. In general, $\gamma(t) = \mathrm{im}\gamma \cap \mathbb{S}_t^{n-1}$, where $\mathbb{S}_t^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = t\}$.

Normally Embedded Sets

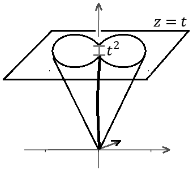
Order of Contact: Given two distinct arcs $\gamma_1, \gamma_2 \subset X$, we define the order of contact (or the tangency order) on the following metrics:

1. **Outer**: $tord(\gamma_1\gamma_2) = \alpha$, where $\|\gamma_1(t) - \gamma_2(t)\| = ct^\alpha + o(t^\alpha)(c > 0)$
2. **Inner**: $tord_X(\gamma_1, \gamma_2) = \beta$, where $d_X(\gamma_1(t), \gamma_2(t)) = ct^\beta + o(t^\beta)(c > 0)$

Examples of non-LNE set germs

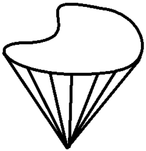


The cusp $y^2 = x^3$.

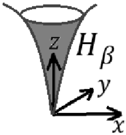


The germ, at the origin, of the set
$\{(x,y,t) \in \mathbb{R}^3 : t \geq 0, \left((x-t)^2 + y^2 - t^2\right)\left((x+t)^2 + y^2 - t^2\right) = t^8\}$

Examples of LNE set germs



Any cone of a smooth set



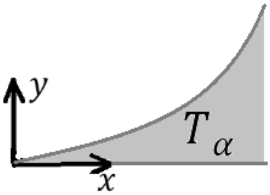
The standard $\beta - horn$

The Inner Bi-Lipschitz Classification

The inner bi-Lipschitz classification problem is well understood. In fact, we have a complete local classification of closed semialgebraic surfaces in (BIRBRAIR, 1999) and a complete global classification for semiagebraic surfaces with isolated inner Lipschitz singularities (FERNANDES, SAMPAIO, 2022).

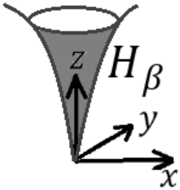
In order to understand the main objects of this (and further) classification and its relation with knot theory, one needs to know the concepts of Hölder triangles and horns.

The Inner Bi-Lipschitz Classification



$T_\alpha := \alpha -$ **Hölder Triangle**

$\{(x,y)\in\mathbb{R}^2 \mid 0 \le x \le 1; 0 \le y \le x^\alpha\}$



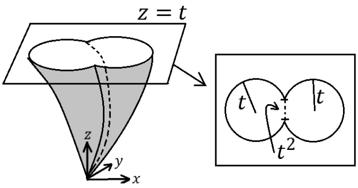
$H_\beta := \beta -$ **Horn**

$\{(x,y,t)\in\mathbb{R}^3 \mid x^2 + y^2 = t^{2\beta}\}$

By (BIRBRAIR, 1999), the exponents α and β are inner bi-Lipschitz invariants in surface germs.

The Outer Bi-Lipschitz Classification

The outer bi-Lipschitz classification of is a (much) harder problem. One reason is due to metric obstructions in arcs, as we can see below.



The study of such obstructions is object of intense research. See, for instance, (BIRBRAIR, BRASSELET,2000), (BOBADILLA, PE PEREIRA, HEINZE, SAMPAIO, 2019), (GABRIELOV, SOUZA, 2022) and others.

The Ambient Bi-Lipschitz Classification

The ambient bi-Lipschitz classification problem is the hardest one. One of the first relevant results is in (SAMPAIO, 2016), and it states that if $(X,0),(Y,0) \subset (\mathbb{R}^n,0)$ are subanaytic sets that are ambient bi-Lipschitz equivalent, then their tangent cones $(C(X),0),(C(Y),0)$ are also ambient bi-Lipschitz equivalent.

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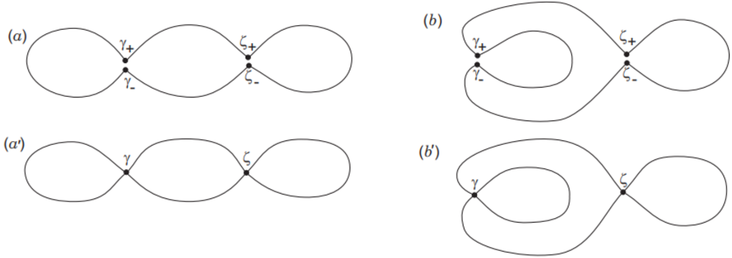
The ambient Lipschitz classification of complex plane curves was solved (NEUMANN, PICHON, 2014), and it shows that topological equivalence implies ambient Lipschitz equivalence.

The Ambient Bi-Lipschitz Classification

In the real case, it is not enough to have same topology and outer Lischitz geometry. Indeed, there are many examples of pairs of semialgebraic surface germs in $\mathbb{R}^3$ and $\mathbb{R}^4$, with the same ambient topology, same inner and outer metric structure, but which are not ambient Lipschitz equivalent.



Ambient Lipschitz Classification of Surface Germs

Since LNE sets are more rigid, it is natural to ask...

The LNE classification problem: Let $(X, 0)$ and $(Y, 0)$ be two LNE surface germs on $(\mathbb{R}^n, 0)$ that are outer bi-Lipschitz equivalent and ambient topologically equivalent. Is it true that $(X, 0)$ and $(Y, 0)$ are ambient bi-Lipschitz equivalent?

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For $n \geq 5$, the answer is **yes**! In fact, since $5 = 2 \times 2 + 1$, you can "unknot" the link by a Lipschitz version of the Whitney embedding theorem in (BIRBRAIR, FERNANDES, JELONEK, 2020) and the result follows.

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But what can we say for the case $n = 3, 4$?

Results in $\mathbb{R}^3$

The main difficulty to obtain Ambient bi-Lipschitz maps consists mainly in three problems.

1. How one can construct bi-Lipschitz maps "locally" on the link?
2. How one can "glue" such maps in order to obtain an ambient bi-Lipschitz map?
3. What is the canonical object that describes your result?

Results in $\mathbb{R}^3$

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3. What is the canonical object that describes your result?

Since the tools developed to tackle this problem in $\mathbb{R}^3$ are very technical, we will first see the results obtained in (MEDEIROS, BIRBRAIR, 2023). In the end of the talk, we present briefly such techniques.

Results in $\mathbb{R}^3$: Isolated Singularity

For surface germs with isolated singularity and connected link, the LNE classification problem have a positive answer. In fact, we have the following.

Theorem (Birbrair, –, 2023)

Let $(X,0),(Y,0) \subset (\mathbb{R}^3,0)$ *be normally embedded semi-algebraic surface germs with links homeomorphic to* $[0,1]$ *or* $\mathbb{S}^1$*. Then,* $(X,0)$ *and* $(Y,0)$ *are ambient bi-Lipschitz equivalent if and only if* $(X,0)$ *and* $(Y,0)$ *are inner bi-Lipschitz equivalent.*

Results in $\mathbb{R}^3$: Isolated Singularity

When the surface have disconnect link, we can deal with each component individually, by the LNE property. This implies a classification theorem, but one must consider how such links are positioned in $\mathbb{S}^2$, and the answer is negative.

Results in $\mathbb{R}^3$: Isolated Singularity

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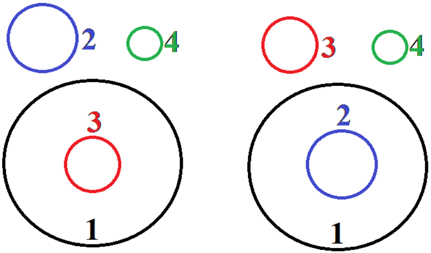


The Non-Isolated Singularity Case

For the non-isolated singularity one must expect that in LNE case, there is a combinatorial data analogue to Hölder complexes, but to find such invariant is still a open problem. For the connected link case it is solved (in preparation).

Theorem (Birbrair, –, 2025)

Let $(X,0),(Y,0) \subset (\mathbb{R}^3,0)$ be LNE semi-algebraic surface germs with connected link. Then, $(X,0)$ and $(Y,0)$ are ambient bi-Lipschitz equivalent if and only if their planar Hölder complexes are isomorphic.

The Case $\mathbb{R}^4$: Introducting Microknots

For LNE surface germs with isolated singularity and connected link, one expects, like the $\mathbb{R}^3$ case, that ambient topology together with outer Lipschitz equivalence implies ambient bi-Lipschitz equivalence.

But, surprisingly, that's false! In order to understand how the counterexamples work, one must see the definition of **microknots.**

The Case $\mathbb{R}^4$: Introducting Microknots

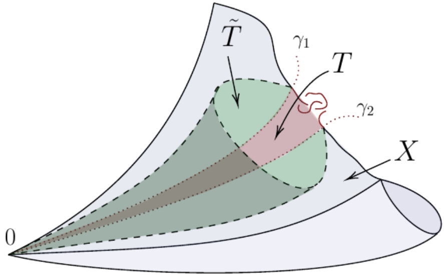
Definition

Let $\beta \in \mathbb{Q}_{\geq 1}$, K be a knot and $X \subset \mathbb{R}^4$ be a LNE surface germ. We say that X contains a β-microknot, corresponding to a knot K, if the following conditions holds:

1. *there are arcs $\gamma_1, \gamma_2 \subset X$ and a β-Hölder triangle $T \subset X$, with γ_1, γ_2 as its boundary arcs;*
2. *there is an ambient trivial 1-Hölder triangle $\tilde{T}' \subset \mathbb{R}^4$, with γ_1, γ_2 as its boundary arcs, and Link$(T \cup T')$ is isotopic to K.*

The Case $\mathbb{R}^4$: Introducting Microknots

Here you can see a figure of a surface germ containing a microknot.



Hornifications

Definition

Let $\eta \in (0, +\infty)$, $\beta \in \mathbb{F}_{\geq 1}$ and K be a knot. By Akbulut-King Theorem, there is a smooth algebraic realization of K. Let $\delta_K : [0, 2\pi] \to \mathbb{S}^3$ be such a map ($\delta_K(0) = \delta_K(2\pi)$) and $\tilde{X}_K \subset \mathbb{S}^3$ be the image of δ_K. Let also $\gamma \in \mathbb{R}^4$ be a straight line and suppose that $\gamma \cap \tilde{X}_K = \emptyset$ and $\tilde{X}_K \subset \mathcal{H}_{\beta,\eta}(\gamma) \cap \mathbb{S}^3$. Consider the following $\mathbb{R}^*_+$-action on $H_{\beta,\eta}(\gamma)$:

$$\Theta_\beta(t, x) = (t, t^\beta \rho(x), v(x))$$

The β-hornification of $\tilde{X}_K$ along γ is the image germ $\bar{X}_{\beta,K} = \Theta_\beta(\tilde{X}_K)$.

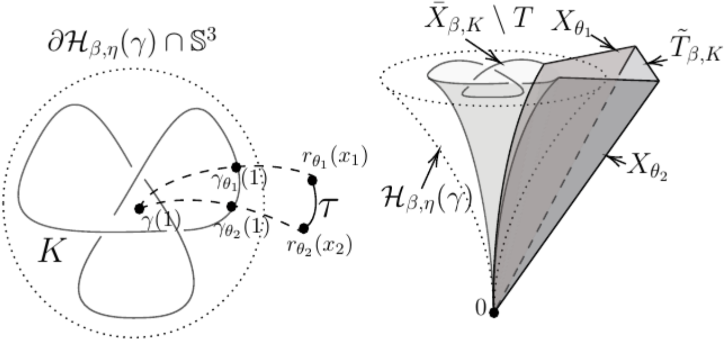
Universality Theorem

Theorem (Birbrair, Denkowski, –, Sampaio, 2024)

For any knot K and $\beta \in \mathbb{F}_{>1}$, there exists an LNE Hölder triangle $T_{\beta,K} \subset \mathbb{R}^4$, containing a β-microknot, isotopic to K. Moreover, two triangles T_{β,K_1} and T_{β,K_2} are ambient bi-Lipschitz equivalent only if K_1 and K_2 are isotopic.

Universality Theorem

Idea of the Proof



Universality Theorem

As a consequence, we obtain:

Theorem (Birbrair, Denkowski, –, Sampaio, 2024)

For any non-trivial knot K there exists two LNE surfaces $Y_K, \tilde{Y}_K \in \mathbb{R}^4$, with isolated singularity at 0, such that

1. *Y_K and $\tilde{Y}_K$ are outer bi-Lipschitz equivalent;*
2. *Y_K and $\tilde{Y}_K$ are ambient topologically equivalent;*
3. *Y_K and $\tilde{Y}_K$ are not ambient bi-Lipschitz equivalent.*

Synchronized Triangles

In what follows, for any $a > 0$, let
$C_a^{n+1} = \{(x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1} \mid t \geq 0;\ x_1^2 + \cdots + x_n^2 \leq (at)^2\}$.

Definition

Let $(\gamma_0, 0), (\gamma_1, 0) \subseteq (C_a^3, 0)$ distinct curve germs and $(T, 0) \subseteq (C_a^3, 0)$ a triangle germ (boundary arcs $(\gamma_0, 0), (\gamma_1, 0)$). We say that $(T, 0)$ is a ***Synchronized Triangle Germ*** *if for every small t, we have $x_0(t) < x_1(t)$ and:*

- $\gamma_0 = \gamma_0(t) = (x_0(t), y_0(t), t); \gamma_1 = \gamma_1(t) = (x_1(t), y_1(t), t)$
- $\pi_z(T \cap \{z = t\})$ *is graph of a semialgebraic function* $f_t : [x_0(t), x_1(t)] \to \mathbb{R}$ *with* $f_t(x_i(t)) = y_i(t)$ *($i = 1, 2$).*

The family of functions $\{f_t\}_{0<t<\varepsilon}$ is called ***Generator of the Synchronized Germ*** $(T, 0)$.

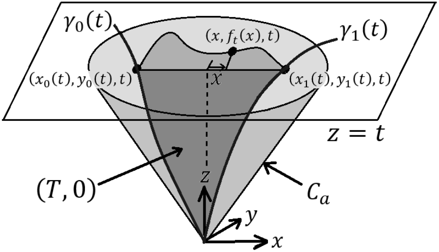
Synchronized Triangles

Arc Coordinate System of (T,0):

Arcs $\gamma_u \subset T$ $(0 \leq u \leq 1)$, where:

$$\theta_u(t) = u \cdot x_1(t) + (1-u) \cdot x_0(t) \in [x_0(t), x_1(t)]$$

$$\gamma_u(t) = (\theta_u(t), f_t(\theta_u(t)), t) \ ; \ t > 0$$



Curvilinear Rectangles

Definition

Let $(T_1,0),(T_2,0) \subset (C_a^3,0)$ be synchronized triangle germs. We say that $(T_1,0),(T_2,0)$ ***are aligned on boundary arcs*** *if there are distinct curve germs $(\gamma_0,0),(\gamma_1,0),(\rho_0,0),(\rho_1,0) \subseteq (C_a^3,0)$ such that, for $i=0,1$, $(\gamma_i,0)$ and $(\rho_i,0)$ are the boundary arcs of $(T_1,0),(T_2,0)$, respectively, and:*

$$\gamma_i = \gamma_i(t) = (x_i(t), y_i(t), t) \ ; \ \rho_i = \rho_i(t) = (x_i(t), w_i(t), t)$$

Let $\{f_t\},\{g_t\}$ are the families of generating functions of $(T_1,0),(T_2,0)$, respectively (WLOG $g_t \leq f_t$) . We define the ***curvilinear rectangle delimited by*** *$(T_1,0),(T_2,0)$ as the germ of:*

$$R = \{(x,y,t) \in C_a^3 \ | \ x_0(t) \leq x \leq x_1(t) \ ; \ g_t(x) \leq y \leq f_t(x)\}$$

If $(\gamma_i,0)=(\rho_i,0)$ such a rectangle is called ***region delimited by*** *$(T_1,0),(T_2,0)$.*

Curvilinear Rectangles

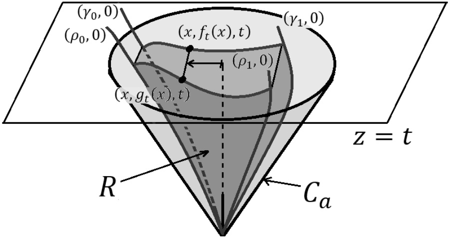
Arc Coordinate System of (R,0):

Arcs $\gamma_{u,v} \subset R$ $(0 \leq u, v \leq 1)$, where:

$$\theta_u(t) = u \cdot x_1(t) + (1-u) \cdot x_0(t) \in [x_0(t), x_1(t)]$$

$$\sigma_{u,v}(t) = v \cdot f_t(\theta_u(t)) + (1-v) \cdot g_t(\theta_u(t)) \in [g_t(\theta_u(t)), f_t(\theta_u(t))]$$

$$\gamma_{u,v}(t) = (\theta_u(t), \sigma_{u,v}(t), t)\ ;\ \ t > 0$$



As a consequence of L-cells decomposition (KURDYKA, ORRO, 1997), we have:

Proposition (δ-Convex decomposition)

Let $a > 0$ and let $X \subset C_a^3$ be a semialgebraic closed 2-dimensional surface. For every $\delta > 0$, $(X, 0)$ can be decomposed in surfaces that are rotations of synchronized triangles $(X_i, 0)$, such that:

1. *If $\{f_t\}$ is the family of generating functions of $(T_i, 0) = (r_{\theta_i}(X_i), 0)$, then each $f_t : [x_0(t), x_1(t)] \to \mathbb{R}$ is a convex function;*
2. *For every $t > 0$ small enough, we have*

$$\left| \frac{\partial f_t}{\partial x_+}(x_0(t)) - \frac{\partial f_t}{\partial x_-}(x_1(t)) \right| < \delta$$

Ambient Bi-Lipschitz Isotopy

Definition

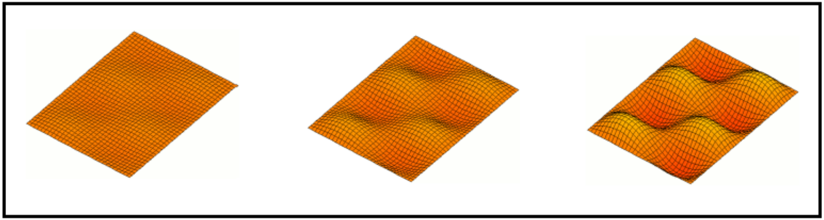
*Let $X, X_0, X_1 \subseteq \mathbb{R}^n$ sets such that $X_1, X_2 \subseteq X$. We say that X_1, X_2 are **Ambient Bi-Lipschitz Isotopic in** X if there is a continuous map $\varphi : X \times [0,1] \to X$ such that, if we denote $\varphi_\tau(p) = \varphi(p, \tau)$, then:*

1. $\varphi_\tau : X \to X$ *is a bi-Lipschitz map (with respect to the induced metric of* $\mathbb{R}^n$*), for all* $0 \leq \tau \leq 1$.
2. $\varphi_0 = id_X$.
3. $\varphi_1(X_1) = X_2$.

Ambient Bi-Lipschitz Isotopy

Definition

The map φ is called ***Ambient Bi-Lipschitz Isotopy in X, taking X_1 into X_2****. We also say that the isotopy φ is* ***Invariant on the Boundary of X*** *if $\varphi_\tau|_{\partial X} = id_{\partial X}$, for all $0 \leq \tau \leq 1$.*



Theorem (Ambient bi-Lipschitz Isotopy in Curvilinear Rectangles)

Let:

$$(T_1,0),(T_2,0),(W_1,0),(W_2,0)\subset(C_a^3,0)$$

be germs of synchronized triangles, two by two alligned on the boundary arcs. If for all $t>0$ small, there is $M>1$ such that:

- $(T_1,0),(T_2,0),(W_1,0),(W_2,0)$ *have M-bounded derivative and that $\{f_t\},\{g_t\},\{a_t\},\{b_t\}$ are their respective families of generating functions;*
- $(T_1,0)$ *has $(\gamma_0,0),(\gamma_1,0)$ as boundary arcs, where:*

$$\gamma_0=\gamma_0(t)=(x_0(t),y_0(t),t)\ ;\ \gamma_1=\gamma_1(t)=(x_1(t),y_1(t),t)$$

and $x_0(t)<x_1(t)$;

Ambient Bi-Lipschitz Isotopy

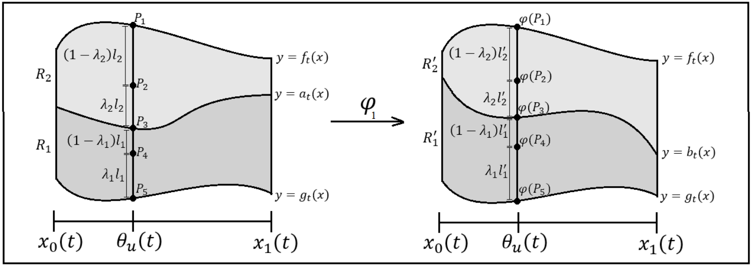
Theorem

- *for all* $x \in (x_0(t), x_1(t))$*, the inequalities are satisfied:*

$$g_t(x) < a_t(x), b_t(x) < f_t(x)$$

$$\frac{1}{M} \leq \frac{a_t(x) - g_t(x)}{f_t(x) - g_t(x)}, \frac{b_t(x) - g_t(x)}{f_t(x) - g_t(x)} \leq 1 - \frac{1}{M}$$

If $(R, 0)$ *is the curvilinear rectangle delimited by* $(T_1, 0), (T_2, 0)$*, then there is a ambient isotopy in* $(R, 0)$*, taking* $(W_1, 0)$ *into* $(W_2, 0)$*. Moreover, if* $(R, 0)$ *is a region, then this ambient isotopy is invariant on the boundary.*

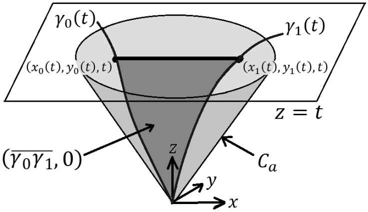


The proof of this theorem is basically a extension to what was done for the $\mathcal{K}$-bi-Lipschitz equivalence criterion (BIRBRAIR, COSTA, FERNANDES, RUAS, 2007) and the $\mathcal{C}^p$-parametrization in o-minimal structures (KOCEL-CYNK, PAWLUCKO, VALLETE, 2019).

Kneadable Triangles

Let $a > 0$, $\gamma_1, \gamma_2 \subset C_a^3$ be two arcs satisfying $tord(\gamma_1, \gamma_2) \neq \infty$, with $\gamma_i(t) = (x_i(t), y_i(t), t)$ $(i = 1.2)$, for every $t > 0$ small enough. We define the **Linear Triangle Delimited by** γ_1, γ_2 as the germ at the origin of the set:

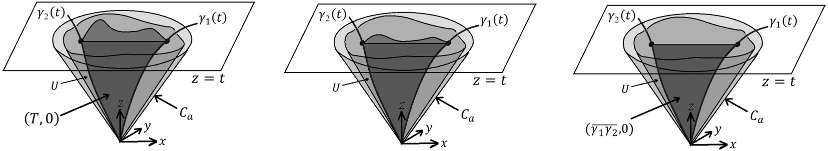
$$\overline{\gamma_1\gamma_2} = \{\lambda\gamma_1(t) + (1-\lambda)\gamma_2(t) \mid t > 0\ ;\ 0 \leq \lambda \leq 1\}$$



Kneadable Triangles

Definition

Let $(T,0) \subset (C_a^3, 0)$ *be a triangle with main vertex at the origin,* γ_1, γ_2 *its boundary arcs and* $(U,0)$ *be a germ of a closed set containing* $(T,0)$*. We say that* $(T,0)$ *is* ***kneadable in*** $(U,0)$ *if there is an ambient bi-Lipschitz isotopy in* U *that takes* $(T,0)$ *into* $(\overline{\gamma_1\gamma_2}, 0)$*, invariant on the boundary of* $(U,0)$*.*



Kneadable Triangles

Proposition

1. *Let* $a > 0$, $\gamma_1, \gamma_2 \subset C_a^3$ *be two arcs such that* $tord(\gamma_1, \gamma_2) = \alpha \neq \infty$. *Then* $(\overline{\gamma_1\gamma_2}, 0)$ *is ambient bi-Lipschitz equivalent to the standard* α*-Hölder's triangle embedded in* $\mathbb{R}^3$.
2. *Let* $(\gamma_i, 0) \in (C_a^3, 0)$ $(i = 1, 2, 3)$ *be distinct arcs such that:*

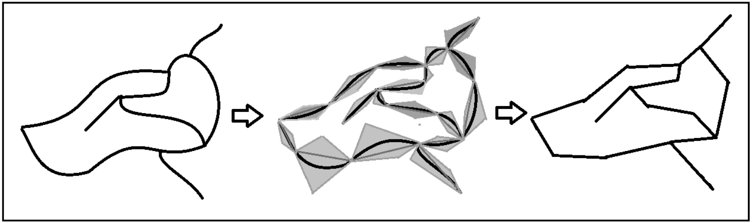
$$(X, 0) = (\overline{\gamma_1\gamma_2} \cup \overline{\gamma_2\gamma_3} \cup \overline{\gamma_3\gamma_1}, 0)$$

is LNE. Then, there exists $\beta \in \mathbb{Q}_{\geq 1}$ *such that* $(X, 0)$ *is ambient bi-Lipschitz equivalent to the germ of the standard* β*-horn.*

Kneadable Triangles

Proposition

Let $a > 0$ and let $(X, 0) \subset (C_a^3, 0)$ be a pure, closed, semi-algebraic, 2-dimensional LNE surface germ with connected link. Then, $(X, 0)$ is ambient bi-Lipschitz equivalent to a germ of a surface formed by a finite union of linear triangles delimited by arcs.



Proof of the Theorem in $\mathbb{R}^3$

Theorem

Let $(X,0),(Y,0) \subset (\mathbb{R}^3,0)$ be normally embedded semi-algebraic surface germs whose link is homeomorphic to $[0,1]$ or $\mathbb{S}^1$. Then, $(X,0)$ and $(Y,0)$ are ambient bi-Lipschitz equivalent if and only if $(X,0)$ and $(Y,0)$ are inner bi-Lipschitz equivalent.

Sketch of the Proof

By [MENDES, SAMPAIO, 2023], since X is LNE, it's link is uniformly C-LNE. Therefore, it's enough to construct the ambient bi-Lipschitz map in the link.

Sketch of the Proof

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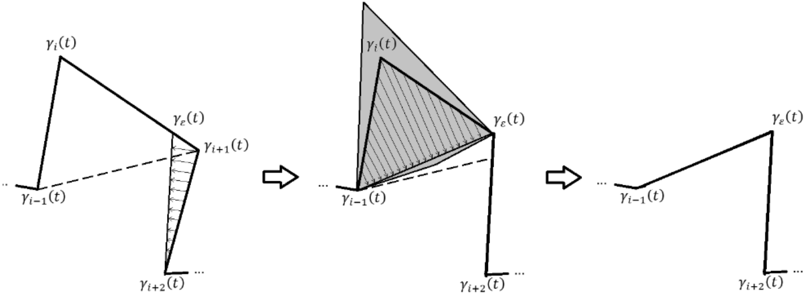
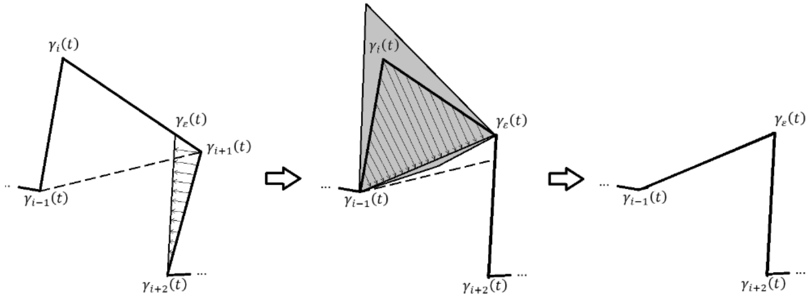
It's also enough to consider X, Y as the union of linear triangles. Now, we reduce the number of such triangles by kneading consecutive triangles into one.

Sketch of the Proof



Sketch of the Proof



In the end, we obtain precisely one Hölder triangle or three Hölder triangles whose link is $\mathbb{S}^1$. The result follows by the inner classification of surfaces in (BIRBRAIR,99).

Some Open Questions

1. If $(X, 0) \subset (\mathbb{R}^3, 0)$ is LNE, have non-isolated singularity and disconnected link, how one can describe their ambient Lipschitz Geometry by a combinatorial object?
2. If $(X, 0) \subset (\mathbb{R}^3, 0)$ is not necessarily LNE, does we have a finite number of ambient Lipschitz equivalence classes to $(X, 0)$, with the same outer Lipschitz geometry and the same ambient topology?
3. $(X, 0) \subset (\mathbb{R}^4, 0)$ is LNE and with isolated singularity, if we "add" microknots with different exponents, does this alter the ambient Lipschitz geometry of them? How those microknots determine such ambient classes?

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Thank You!!!