Differential Field Arithmetic

David Meretzky
University of Notre Dame

Purdue Model Theory and Applications Seminar
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Outline

1. Model theory of differential fields

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2. The Picard-Vessiot differential Galois theory

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3. History of existence theorems for Picard-Vessiot extensions and an extended Galois correspondence

4. History of model-theoretic differential Galois theory and an extended Galois correspondence

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5. Results classifying strictly transitive definable group actions

6. Contemporary model-theoretic approach to differential Galois theory

6. Contemporary existence theorems for differential Galois theory and (differential) field arithmetic

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Definition

An ordinary differential field is a pair (K, δ) where K is a field and $\delta : K \to K$ is a function such that $\forall x, y \in K$

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A differential field arithmetic property guarantees existence of solutions to of certain families of differential equations (over said differential field). For example, being differentially closed.

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Every (K, δ) is contained in $(K^{\text{diff}}, \delta)$, a differential closure (a prime model of DCF_0) with $C_{K^{\text{diff}}} = (C_K)^{\text{alg}}$.

Picard-Vessiot Theory

Let (K, δ) be a differential field. An ordinary linear homogeneous differential equation (OHDLE) over K is an equation of the form

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Ex. Let $K = (\mathbb{Q}, \frac{d}{dt}) \subset (\mathbb{Q}(e^t), \frac{d}{dt}) = L$. Then e^t is a $\mathbb{Q}$-basis of $V(L) = \{ce^t : c \in C_L = C_K = \mathbb{Q}\}$ for $\delta(y) - y = 0$.

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Since $C_{K^{\mathrm{diff}}} = (C_K)^{\mathrm{alg}}$, then $C_L = C_K$ for any intermediate L:
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Existence and uniqueness/multiplicity of Picard-Vessiot extensions is sensitive to field-arithmetic, model-theoretic, and Galois-cohomological properties of C_K.

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4. (Torsor theorem) The realizations of $\text{tp}(\bar{b}/K(C_{K^{\text{diff}}})) = \text{tp}(\bar{b}/K)$ in K^{diff}, $Q_{\bar{b}}(K^{\text{diff}})$, is a right K-definable torsor for $G(C_{K^{\text{diff}}})$.

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(Epstein 1955): One can always find a fundamental system $\bar{b}$ for Y, an OHLDE, such that C_L/C_K is a Galois extension.

PV extensions with new algebraic constants (Ch 4.)

Theorem 4.3.4, Generalizes Epstein '55 Theorem 1.

Let L/K be generated by a fundamental system with C_L/C_K a finite algebraic extension. Then $\text{Aut}(L/K) \cong G(C_K)$ for G a linear algebraic group defined over C_K.

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Theorem 4.6.1, Model theoretic proof of E. '55 Theorem 9.

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Proposition 4.5.10.

Moreover F' is fixed under the direct correspondence if and only if $H_{F'}(C_L)$ is Zariski dense in $H_{F'}$.

Totally Transcendental Theories (Ch. 1.)

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Poizat remarks in 1978 that because for $T = \mathrm{DCF}_0$, $A = K$, $M = K^{\mathrm{diff}}$, $K \subseteq K^{PV} \subseteq \mathrm{mcl}^{\mathrm{eq}}(K) \subset K^{\mathrm{diff}}$, it is worthwhile to study $\mathrm{Aut}(\mathrm{mcl}^{\mathrm{eq}}(A)/A)$.

Galois correspondence for the minimal closure (Ch. 1.)

Theorem 1.4.20, joint with Anand Pillay.

Let T be a totally transcendental theory, A a set of parameters and M a prime model over A. There is a Galois correspondence

$$\{\text{definable } H \subseteq \mathrm{Aut}(\mathrm{mcl}^{\mathrm{eq}}(A)/A)\} \leftrightarrow \{B : A \subseteq B \subseteq \mathrm{mcl}^{\mathrm{eq}}(A)\}$$

with definable meaning $H = \{\sigma \in \mathrm{Aut}(\mathrm{mcl}^{\mathrm{eq}}(A)/A) : \varphi(\bar{b}, \sigma(\bar{b}))\}$ for some A-definable $\varphi(\bar{x}, \bar{y})$ and tuple $\bar{b}$ from $\mathrm{mcl}^{\mathrm{eq}}(A)$ and with B finitely dcl-generated and dcl-closed.

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Motivations: 1) K^{diff} does not have good enough homogeneity properties. 2) Make explicit/treat model theoretically more recent work Magid (2022).

Timing: Poizat was missing this notion of definable subgroup. It was adapted by Pillay in (2024) from Hrushovski Krupinski Pillay (2021).

Definable Galois cohomology (Ch 2.)

Let T be a totally transcendental theory with EI. An intermediate set $A \subseteq B \subseteq M$ is normal if $\mathrm{Aut}(M/B)$ is normal in $\mathrm{Aut}(M/A)$. Again assume M is prime over A. Let G be an A-definable group and P a free and transitive A-definable right G-set (PHS). The following definitions are adapted from Pillay (1997).

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Definition 2.2.4, M. (2025)

A map $\varphi : \mathrm{Aut}(B/A) \to G(B)$ is a definable cocycle if $\forall \sigma, \tau \in \mathrm{Aut}(B/A)$,

$$\varphi(\sigma\tau) = \varphi(\sigma)\sigma(\varphi(\tau))$$

and there exists an A-definable function $h(\bar{x}, \bar{y})$ and $\bar{b} \in B^n$ such that

$$\varphi(\sigma) = h(\bar{b}, \sigma(\bar{b})).$$

Two definable cocycles φ and ψ are cohomologous if $\exists g \in G(B)$ with

$$\varphi(\sigma) = g^{-1}\psi(\sigma)\sigma(g).$$

Definable Galois cohomology (Ch 2.)

Definition 2.2.7, Proposition 2.2.12, M. (2025)

The set of cocycles is denoted $\mathrm{Z}^1_{\mathrm{def}}(B/A, G(B))$ and is in bijection with $\mathrm{P}_{\mathrm{def},*}(M/A, G(B))$ the A-definable isomorphism classes of PHSs for G with a specified B-point.

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where the last term denotes the fixed points of an adaptation of Serre's transform action of $\mathrm{Aut}(B/A)$ on $\mathrm{H}^1_{\mathrm{def}}(M/B, G(M))$.

LES in definable Galois cohomology (Ch. 2)

Theorem 2.4.1, Corollary 2.4.20, León Sánchez, M., Pillay (2024)

A short exact sequence of A-definable groups, $1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$,

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A short exact sequence of A-definable groups, $1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$, induces a long exact sequence in cohomology:

$$1 \to N(A) \xrightarrow{\iota^0} G(A) \xrightarrow{\pi^0} Q(A) \xrightarrow{\delta}$$
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Moreover, the cardinality of $\mathrm{H}^1_{\mathrm{def}}(M/A, G(M))$ is bounded by the cardinality of the union of $\mathrm{H}^1_{\mathrm{def}}(M/A, Q(M))$ and all $\mathrm{H}^1_{\mathrm{def}}(M/A, {}_{P_\alpha}N(M))$ as ${}_{P_\alpha}N(M)$ ranges over a set of PHSs $\{P_\alpha\}$ representing the fibers of π^1.

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These sequences are the main tools for computing with definable Galois cohomology.

Internality in DCF_0 (Ch. 3)

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We say Y is internal to X if $Y \subseteq \mathrm{dcl}(K, X, B)$ for some small set of parameters B. By compactness and a standard coding trick there is a K-definable function and a tuple $\bar{b}$ from B^n such that $f(\bar{b}, \bar{x}) : X^n \to Y$ is surjective.

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The set of such fundamental systems Z is K-definable. Following some replacements in notation $f(b, \bar{x}) : X_1 \to Z$ is a bijection for any $b \in Z$. $X_1 \subseteq (X^n)^{\mathrm{eq}}$.

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We can define a definable groupoid action living on Z and X_1 as follows.

Hrushovski (2004): Binding groupoid (Ch. 3)

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For any distinct types p_1 and p_2 of fund. sys. over $\mathrm{dcl}(K, X)$, and associated tuples $\bar{d}_1$, $\bar{d}_2$, we define

$$H_{\bar{d}_1, \bar{d}_2} = \{c \in X_1 : f(b_1, c) = b_2,\ b_1 \models p_1,\ b_2 \models p_2\}$$

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Additionally, there is a K-definable group H^+ acting on the left on each $Q_{\bar{d}}$ isomorphically to the action of $\mathrm{Aut}(Q_{\bar{d}}/\mathrm{dcl}(K, X))$. Giving each triple $(H^+, Q_{\bar{d}}, H_{\bar{d}})$ the structure of a definable biPHS.

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Elements $b \in Q_{\bar{d}}(K^{\mathrm{diff}})$, by the definitions, generate generalized strongly normal extensions exactly when $\bar{d} \in X(K)$. Then both H^+ and $H_{\bar{d}}$ are K-definable and are the intrinsic and extrinsic differential Galois groups of the extension L/K where $L = \mathrm{dcl}(K, b) = K\langle b\rangle$.

Cohomology and extensions (Ch. 3)

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Proof.

For any other $b_1 \in Z(K^{\mathrm{diff}})$, with $\bar{d}_1 \in X(C_K)$, then $H_{\bar{d}_1,\bar{d}_0}(C_{K^{\mathrm{diff}}})$ is a right C_K-definable PHS.

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Cohomology and extensions (Ch. 3)

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Then $P(C_{K^{\text{diff}}}) = H_{d_1 d_0}(C_{K^{\text{diff}}})$ for some $\bar{d}_1 \in (C_K)^n$:

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So $P(C_{K^{\text{diff}}})$ is a C_K-definable right coset of $H_{\bar{d}_0}(C_{K^{\text{diff}}})$ in $GL_n(C_{K^{\text{diff}}})$.
But $Z(K^{\text{diff}})$ is a right K-definable PHS for $GL_n(C_{K^{\text{diff}}})$.
Then $P(C_{K^{\text{diff}}}) = H_{d_1 d_0}(C_{K^{\text{diff}}})$ for some $\bar{d}_1 \in (C_K)^n$:Take an element C in this coset. Apply the inverse C^{-1} to b_0 to get a new solution b_1 associated to $\bar{d}_1 \in (C_K)^n$.

Cohomology and extensions (Ch. 3)

proof.

Now let $P(C_{K^{\mathrm{diff}}})$ be a right C_K-definable PHS for $H_{\bar{d}_0}(C_{K^{\mathrm{diff}}})$.
As $H_{\bar{d}_0}(C_{K^{\mathrm{diff}}})$ is linear, we apply the LES

$$1 \to H_{\bar{d}_0}(C_K) \to GL_n(C_K) \to (GL_n/H_{\bar{d}_0})(C_K) \xrightarrow{\delta^1} \mathrm{H}^1_{\mathrm{alg}}(C_{K^{\mathrm{diff}}}/C_K, H_{\bar{d}_0}(C_{K^{\mathrm{diff}}})) \to 1 \; (= \mathrm{H}^1_{\mathrm{alg}}(C_{K^{\mathrm{diff}}}/C_K, GL_n(C_{K^{\mathrm{diff}}})))$$

So $P(C_{K^{\mathrm{diff}}})$ is a C_K-definable right coset of $H_{\bar{d}_0}(C_{K^{\mathrm{diff}}})$ in $GL_n(C_{K^{\mathrm{diff}}})$.
But $Z(K^{\mathrm{diff}})$ is a right K-definable PHS for $GL_n(C_{K^{\mathrm{diff}}})$.
Then $P(C_{K^{\mathrm{diff}}}) = H_{d_1 d_0}(C_{K^{\mathrm{diff}}})$ for some $\bar{d}_1 \in (C_K)^n$:Take an element C in this coset. Apply the inverse C^{-1} to b_0 to get a new solution b_1 associated to $\bar{d}_1 \in (C_K)^n$. So the map is surjective.

Existence theorems and countability results (Ch. 3)

Ovchinnikov, Gorchinsky, Gillet, (2013), Kamensky, Pillay (2015): If C_K is existentially closed in K as a field then PV extensions exist for any OHLDE Y over K in K^{diff}.

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Existence theorems and countability results (Ch. 3)

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Kamensky, Pillay (2015): If C_K is existentially closed in K as a field, and C_K is bounded and large as a field then PV extensions L exist for any OHLDE Y over K in K^{diff} so that C_K is e.c. in L.

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Theorem 3.2.17, Kamensky, Pillay (2015), León Sánchez, M., Pillay (2024)

Suppose the field F is bounded and G is any algebraic group over F. Then $\mathrm{H}^1_{\text{alg}}(F^{\text{alg}}/F, G(F^{\text{alg}}))$ is at most countable.

Countability and Boundedness (Ch. 3)

Theorem 3.4.13. León Sánchez, M., Pillay (2024)

Let (K, δ) be a differentially large field. Furthermore, suppose that K is bounded as a field. Then $\mathrm{H}^1_\delta(K^{\mathrm{diff}}/K, G(K^{\mathrm{diff}}))$ is countable for any differential algebraic group G defined over K.

Countability and Boundedness (Ch. 3)

Theorem 3.4.13. León Sánchez, M., Pillay (2024)

Let (K, δ) be a differentially large field. Furthermore, suppose that K is bounded as a field. Then $\mathrm{H}^1_\delta(K^{\mathrm{diff}}/K, G(K^{\mathrm{diff}}))$ is countable for any differential algebraic group G defined over K.

Pillay (2017) Minchenko, Ovchinnikov (2019): A differential field K is algebraically closed and is closed under PV extensions if and only if $\mathrm{H}^1_{\mathrm{def}}(K^{\mathrm{diff}}/K, G(K^{\mathrm{diff}})) = 1$ for all linear differential algebraic groups G defined over K.

Thank you!

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