Chapter 1

Language and Truth Tables for the Sentential Calculus

In the first four chapters, we develop a formal system called the *Sentential Calculus* and discuss some of its applications. We begin by studying the language of this system. The language is quite simple. It consists of sentences and sentential connectives. We shall use capital letters, $A, B, C, D, \ldots$, and these letters with subscripts, to represent sentences. The connectives we shall use are negation, conjunction, disjunction, the conditional, and the biconditional. The language is not very expressive (one would not write poetry in this language, for example), but, as we shall see, it does have important applications.

We shall be interested both in the *syntax* (structure, grammar) and the *semantics* (meaning) of this language. Both of these concepts are described below.

1.1 NEGATION

Negation is translated as "not" or "it is not the case that." The symbol we shall use is $\neg$. Other symbols that logicians use for negation are: $-$, $\sim$, and $^{-}$. Thus, if S is the sentence "It is snowing outside," $(\neg S)$ is "It is not snowing outside" or "It is not the case that it is snowing outside."

The effect of the symbol $\neg$ on the meaning of a sentence S is illustrated by the following table, which is called a *truth table for* $\neg$:

S	$\neg S$
T	F
F	T

Thus, if the sentence S is true, then $(\neg S)$ is false and if S is false, then $(\neg S)$ is true. (When there is no ambiguity, we shall frequently omit the parentheses from "$(\neg S)$.")

1.2 CONJUNCTION

The translation of "conjunction" is "and" or sometimes "but." The symbol we shall use is $\wedge$. Other symbols used for conjunction include & and $\cdot$. For example, if S is the sentence "It is snowing outside" and W is the sentence "It is warm inside," then $(S \wedge W)$ would be translated as "It is snowing outside and warm inside" or "It is snowing outside, but warm inside."

If, in addition, C is the sentence "It is cold outside," then the compound sentence "It is snowing and cold outside, but warm inside" would be translated into our language as $((S \wedge C) \wedge W)$.

The sentence "It is cold outside, but not snowing," is translated as $(C \wedge \neg S)$. The sentence "It is not the case that it is cold and snowing outside" is translated as $\neg(C \wedge S)$.

The sentence "It is not both cold and snowing outside" would be translated the same way: $\neg(C \wedge S)$.

The following is the truth table for $\wedge$:

A	B	$(A \wedge B)$
T	T	T
T	F	F
F	T	F
F	F	F

A conjunction $(A \wedge B)$ is *true* if and only if both conjuncts, A and B, are true. (We shall sometimes use "iff" as an abbreviation for "if and only if.")

1.3 DISJUNCTION

By "disjunction," in our language, we mean the "nonexclusive or" or "and/or." We use the symbol $\vee$. (Sometimes + is used as the symbol for disjunction.) For example,

Sentence	Translation
1. It is snowing or cold outside.	$(S \vee C)$
2. It is either snowing and cold outside, or warm inside.	$((S \wedge C) \vee W)$
3. It is neither snowing outside nor cold outside.	$\neg(S \vee C)$

Sentence	Translation
4. It is not snowing outside and it is not cold outside.	$(\neg S \wedge \neg C)$
5. It is snowing or cold outside, but not both.	$((S \vee C) \wedge \neg(S \wedge C))$

(Sentence 5 exemplifies how the "exclusive or" is translated in our language.)

The truth table for disjunction is as follows:

A	B	$(A \vee B)$
T	T	T
T	F	T
F	T	T
F	F	F

Thus, a disjunction is *false* iff both disjuncts are false.

The following are truth tables for the translations of sentences 3 and 4:

S	C	$(S \vee C)$	$\neg(S \vee C)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

S	C	$\neg S$	$\neg C$	$(\neg S \wedge \neg C)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

We see that the two formulas have the same truth table; that is, one formula is true if and only if the other is true. (See De Morgan's rules in Chapter 2.)

The truth tables for negation, conjunction, and disjunction show that these connectives are *truth functional.* Given the truth value, T or F, of each component part of a compound sentence containing these connectives, the truth value of the whole sentence is uniquely determined. An example of this is presented in the following truth table for the formula $S = ((A \wedge \neg B) \vee \neg(A \wedge C))$:

A	B	C	$\neg B$	$(A \wedge \neg B)$	$(A \wedge C)$	$\neg(A \wedge C)$	S
T	T	T	F	F	T	F	F
T	T	F	F	F	F	T	T
T	F	T	T	T	T	F	T
T	F	F	T	T	F	T	T
F	T	T	F	F	F	T	T
F	T	F	F	F	F	T	T
F	F	T	T	F	F	T	T
F	F	F	T	F	F	T	T

We see from the truth table that S is false if A, B, and C are each true, and S is true otherwise. (We also note that the truth table for a formula that contains n letters has 2^n lines. Why?)

The truth tables of the remaining connectives will also be defined so they are truth functional because we want truth tables to depend only on the structure of a sentence, not on its meaning.

1.4 THE CONDITIONAL

The normal translation of the conditional is "if · · · , then . . . ." We have various ways of expressing this in English, however, as we shall see below. The symbol for the conditional that we shall use is $\rightarrow$. (Some texts use the symbol $\supset$.) A sentence of the form "$(A \rightarrow B)$" is called a *conditional* statement; "$(B \rightarrow A)$" is called its *converse*; "$(\neg A \rightarrow \neg B)$" its *inverse*; and "$(\neg B \rightarrow \neg A)$" its *contrapositive*. Some of the ways "$(A \rightarrow B)$" is translated include "if A then B," "B if A," "A implies B," "B is a consequence of A," "B provided that A," "A is a sufficient condition for B," and "B is a necessary condition for A."

In the following examples, when we write a sentence followed by letters in parentheses, we mean that those are the letters that are to be used for the *atomic sentences*, those sentences that contain no connectives. We treat atomic sentences as our basic units. (We shall avoid using the letters "T" and "F" for atomic sentences because we use these letters for truth values.)

Sentence	Translation
1. A triangle is isosceles if it has two equal angles. (I, E)	$(E \rightarrow I)$
2. If $x^2 > 4$, then $x > 2$, provided that x is positive. (O, W, P)	$(P \rightarrow (O \rightarrow W))$
3. $x > 7$ implies $x + 1 > 8$, and conversely. (S, E)	$((S \rightarrow E) \wedge (E \rightarrow S))$
4. A necessary condition for two lines to be parallel is that they neither intersect nor coincide. (P, I, C)	$(P \rightarrow \neg(I \vee C))$
5. If certain numbers such as 5 or 6 are substituted for x in the inequality $x^2 - 6 < 2x + 4$, we obtain the false statements $19 < 14$ or $30 < 16$, respectively. (V, S, N, H)	$((V \rightarrow N) \wedge (S \rightarrow H))$

Notice that in some of the sentences given in the preceding table, we had to change the order of the atomic sentences before we could translate them. Sentence 1 was rewritten as "If a triangle has two equal angles, it is isosceles." (Sometimes, in conditional statements, "then" is omitted.) Similarly, sentence 2 was changed to "If x is positive, then if $x^2 > 4$, $x > 2$." Sentence 5 required more extensive changes: "If 5 is substituted for x in the inequality $x^2 - 6 < 2x + 4$, we obtain the false statement $19 < 14$, and if 6 is substituted for x in the inequality $x^2 - 6 < 2x + 4$, we obtain the false statement $30 < 16$."

In the conditional statement "$(A \rightarrow B)$," "A" is called the *antecedent* or *hypothesis*, and "B" is called the *consequent* or *conclusion*. Normally, when such a statement is used in English, there is some connection between the antecedent and consequent. We usually do not say "If $3 > 7$, then two is a prime." In order to make the conditional truth functional, however, we have to decide on a truth value for such a statement. It seems natural that if A and B are both true, then $(A \rightarrow B)$ is true, and if A is true and B is false, then $(A \rightarrow B)$ is false. We shall make the convention, which is used in mathematics, that if A is false, then $(A \rightarrow B)$ is true. Thus, the truth table for $\rightarrow$ is as follows:

A	B	$(A \rightarrow B)$
T	T	T
T	F	F
F	T	T
F	F	T

A conditional statement is false iff the antecedent is true and the consequent is false.

1.5 THE BICONDITIONAL

The symbol we shall use for the biconditional is $\leftrightarrow$. (Another symbol that is frequently used is $\equiv$.) The formula "$(A \leftrightarrow B)$" is translated as: A if and only if B, A is equivalent to B, or A is a necessary and sufficient condition for B. The natural meaning for $(A \leftrightarrow B)$ is that it is true if and only if A and B have the same truth value. That is,

A	B	$(A \leftrightarrow B)$
T	T	T
T	F	F
F	T	F
F	F	T

We would expect that "$(A \leftrightarrow B)$" and "$((A \rightarrow B) \wedge (B \rightarrow A))$" have the same meaning. This is indeed the case, as we see by the following truth table:

A	B	$(A \rightarrow B)$	$(B \rightarrow A)$	$((A \rightarrow B) \wedge (B \rightarrow A))$	$(A \leftrightarrow B)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

This gives additional support for the truth table we choose for the conditional.

Because "$((A \rightarrow B) \wedge (B \rightarrow A))$" and "$(A \leftrightarrow B)$" have the same meaning and "$(B \rightarrow A)$" is translated as "A if B," then another translation of "$(A \rightarrow B)$"

is "*A only if B*." The sentence "Virginia dates Oscar only if Walter is out of town" is interpreted as "If Virginia dates Oscar then Walter is out of town."

Similarly, the following three sentences have different shades of meaning, but are translated the same way in our language:

1. If an animal moos, then it is a cow.
2. An animal moos only if it is a cow.
3. The only animal that moos is a cow.

The connective *unless* is occasionally used in English. In our language, we shall translate "A unless B" as "$(\neg B \rightarrow A)$." The sentence "Oscar will come unless it rains" means the same as "If it does not rain, then Oscar will come." If we make a truth table for the formula "$(\neg B \rightarrow A)$" we see it has exactly the same truth table as "$(A \vee B)$." Thus, In our language, "A unless B" means the same as "A or B." Each of the following four sentences could be translated as "$\neg U \rightarrow R$" (or "$U \vee R$") in the language of the Sentential Calculus:

1. If I do not take an umbrella, then it rains.
2. It rains if I do not take an umbrella.
3. It rains unless I take an umbrella.
4. Either I take an umbrella or it rains.

The following table presents additional examples of translations:

Sentence	**Translation**
1. Oscar will flunk the logic course unless he studies and does his homework. (K, S, H)	$(\neg (S \wedge H) \rightarrow K)$ or $(K \vee (S \wedge H))$
2. A necessary and sufficient condition for two triangles to be similar is that they have equal angles (S, E)	$(S \leftrightarrow E)$
3. Assuming that either logic is difficult or that the text is not readable, Oscar will pass only if he concentrates. (D, R, P, C)	$((D \vee \neg R) \rightarrow (P \rightarrow C))$

Note that the sentence "The text is not readable" is not an atomic sentence, because it contains the connective "not." Also note that translations do not require any knowledge of underlying facts. For example, we do not have to know what sort of student Oscar is or what the definition of similar triangles might be to translate the preceding sentences.

1.6 LANGUAGE OF THE SENTENTIAL CALCULUS

The symbols of the language are as follows:

Sentential variables: $A, B, C, \ldots$, and these letters with subscripts.

Connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

Parentheses: (,) (Parentheses are used for grouping.)

Rules for Formula Construction

The following are rules for formula construction:

1. If A is a sentential variable, then A is a formula.
2. If P and Q are formulas, then each of the following are formulas:

$$(\neg P),\ (P \wedge Q),\ (P \wedge Q),\ (P \rightarrow Q), \text{ and } (P \leftrightarrow Q).$$

3. No other expressions are formulas.

(We shall have more to say about this type of definition, called an *inductive* definition, in Chapter 15. For now, it is sufficient to be able to distinguish a formula from a meaningless string of symbols.)

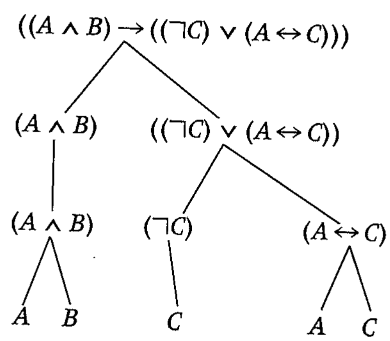
The definition of formulas given above describes how formulas are constructed from sentential variables. For each formula, we can construct a "tree" starting with sentential variables. The following tree for the formula $((A \wedge B) \rightarrow ((\neg C) \vee (A \leftrightarrow C)))$ is given as an example:

(4)

(3)

(2)

(1)



The first level of the tree consists of sentential variables. Each subsequent level is obtained by applying rule 2 to formulas on the preceding level or using formulas already constructed. (The formula "$(A \wedge B)$" on line (2) was repeated on line (3).)

1.7 SEMANTICS

The following truth table describes how we interpret the connectives in our language and thereby give meaning to arbitrary formulas:

P	Q	$\neg P$	$(P \wedge Q)$	$(P \vee Q)$	$(P \rightarrow Q)$	$(P \leftrightarrow Q)$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

We mention the fact that there are connectives in English that are not truth functional. The connective "it is possible that" is an example. The symbol that is used is $\Diamond$. If S is the false statement "$2 + 2 = 5$," then $\Diamond S$ is false. However, if S is the false statement "Someone has lived to be 200 years old," $\Diamond S$ is true. It is possible that someone has lived to be 200 years old. Consequently, the truth value of $\Diamond S$ cannot be determined from the truth value of S alone.

The language we constructed in this chapter is much simpler than English (or any other spoken language). In our language, for example, "and" means the same as "but"; "only if" the same as "then"; and "unless" the same as "or." Also, there are many expressions in English that we do not have in our language. For example, the connectives "it is possible that" and "it is necessary that" are not expressible in the Sentential Calculus. In spite of all these limitations, we shall see in later chapters that the Sentential Calculus has many applications.

1.8 CONVENTIONS FOR PARENTHESES

To reduce the number of parentheses used in formulas, we shall use the following conventions:

1. The outermost parentheses around a formula will be omitted.
2. Negation binds more closely than any other connective.
3. Conjunction and disjunction bind more closely than the conditional and biconditional.

Without the Convention	**With the Convention**
1. $((\neg A) \wedge B)$	$\neg A \wedge B$
(Note the difference between "$\neg A \wedge B$" and "$\neg(A \wedge B)$.")	
2. $((A \wedge (\neg B)) \rightarrow C)$	$A \wedge \neg B \rightarrow C$
3. $((A \wedge B) \vee (\neg C))$	$(A \wedge B) \vee \neg C$
4. $(((\neg A) \vee B) \rightarrow (C \leftrightarrow D))$	$\neg A \vee B \rightarrow (C \leftrightarrow D)$
5. $(\neg(A \rightarrow B) \rightarrow (C \vee D))$	$\neg(A \rightarrow B) \rightarrow C \vee D$

Some sentences without parentheses are ambiguous. For example, "$(A \wedge B) \vee C$" has a different truth table from "$A \wedge (B \vee C)$." Thus, we cannot write "$A \wedge B \vee C$" without any parentheses (unless we introduce additional conventions for parentheses). Parentheses may always be added to a formula to clarify its meaning. In some instances, it might be advantageous to write "$A \land \neg B \to C$" as "$(A \land \neg B) \to C$" to emphasize that "$A \land \neg B$" is the antecedent. In most cases, we shall use the preceding conventions for reducing parentheses.

EXERCISES 1

A. Translate each of the following sentences into the language of the Sentential Calculus using the indicated letters for atomic sentences. (Answers for 5, 7, 11, 15, 17, 20, 22, and 24.)

(1) Oscar and Virginia both go to college. (O, V)
(2) Oscar registered for the logic course, but Virginia did not. (O, V)
(3) Neither Oscar nor Virginia like George. (O, V)
(4) If Oscar dates Virginia, then George does not. (O, G)
(5) Either Oscar dates Virginia or George dates Virginia, but not both. (O, G)
(6) Oscar will pass the logic course only if he studies. (P, S)
(7) Oscar will not pass the logic course unless he does his homework and studies. (P, H, S)
(8) Oscar will not pass the logic course if he neither does his homework nor studies. (P, H, S)
(9) It is not the case that Oscar will pass the logic course provided that he studies and does his homework. (P, S, H)
(10) A sufficient condition that Oscar pass the logic course is that he studies and does his homework. (P, S, H)
(11) If Oscar doesn't study and do his homework, then he will not pass the logic course. (S, H, P)
(12) If Oscar and Virginia work at a steady pace, then there is no gain or loss of efficiency when they work together. (O, V, G, L)
(13) If I miss my train, I will arrive 10 minutes late, assuming the next train is on time. (M, L, N)
(14) We will go to the park today provided that our car doesn't break down and it doesn't rain. (P, C, R)
(15) If logic is difficult, Oscar and Virginia will pass only if they study. (D, O, V, S)
(16) If two lines lie in a plane, a necessary and sufficient condition for them to be parallel is that they neither intersect nor coincide. (L, P, I, C)
(17) If Q is a quadrilateral, then Q is a parallelogram if and only if its opposite sides are both equal and parallel. (Q, P, E, L)
(18) If the function f is continuous on the interval (a, b), then either f has a maximum on $[a, b]$ or f is not continuous at both a and b. (C, M, A, B)
(19) A sufficient condition for the function f to have a maximum on

$[a, b]$ is that f is continuous on (a, b) and f is continuous at both a and b. (M, C, A, B)

(20) If f' is defined on the interval (a, b), a necessary condition for f to be increasing on (a, b) is that f' is positive on (a, b). (D, I, P)

(21) A necessary and sufficient condition for f' to be positive on (a, b) is that f' is defined on (a, b) and f is increasing on (a, b). (P, D, I)

(22) If $I = \int_a^b f(x)\, dx$ and A is the approximation of $\int_a^b f(x)\, dx$ using the trapezoidal rule, then $A \geq I$ if $f''(x) > 0$ for $x \in [a, b]$. (I, A, G, P)

(23) If Oscar's grades on his first two exams were 72 and 95, then his letter grades on these two exams were C and A, respectively. (S, N, C, A)

(24) If 3 and 4 are substituted for x and y, respectively, in the inequality $2x + y < x + 3y$, then we obtain the inequality $10 < 15$. (S_3, S_4, I)

(25) If $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are three vectors in $\mathbb{R}^3$ and their initial points are at the origin, then the set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent if and only if the three vectors lie in the same plane through the origin. (V, I, D, P)

B. Assign letters to the atomic sentences, and translate each of the following sentences into a logical formula. (Answers for 2, 5, 9, and 11.)

(1) If Virginia passed algebra or geometry, then she was admitted into college.

(2) If Virginia did not pass both algebra and geometry, then she would be admitted to neither Harvard nor Yale.

(3) It is not the case that if Oscar secures employment, then he will marry Virginia.

(4) Oscar will look for employment if he flunks out of college, unless he is admitted into a technical school.

(5) Oscar and Virginia will graduate from college only if he/she passes the logic course.

(6) A necessary and sufficient condition for three points A, B, and C to lie on a line is that the distance from A to C is the sum of the distance from A to B and from B to C.

(7) If f is continuous and differentiable on $[a, b]$, then either there is a horizontal tangent line to f between a and b or not both $f(a)$ and $f(b)$ are zero.

(8) A necessary condition for f to be continuous at $x = a$ is that $f(a)$ is defined and $\lim_{x \to a} f(x)$ exists.

(9) If x is an integer, then exactly one of x and $x + 1$ is an even integer.

(10) If f is continuous but not integrable on (a, b), then f is not defined at both a and b.

(11) If A is a false sentence, then the conditional sentence $A \rightarrow B$ is of the form $F \rightarrow T$ or $F \rightarrow F$, depending upon whether B is true or false, respectively.

(12) If f is a function and f has continuous derivatives, $f^{(n)}$, in a neighborhood N of zero, for $n = 0, 1, 2, \ldots$, then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ for $x \in N$ if and only if the remainder,

$$R_{n+1}(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt,$$

approaches zero as n increases without limit.

C. Give the converse, inverse, and contrapositive of each of the following conditional sentences (Answers for 2 and 7.)
   (1) If $\vec{v}$ is parallel to $\vec{w}$, then $|\vec{v} \cdot \vec{w}| = \|\vec{v}\|\,\|\vec{w}\|$.
   (2) Two lines will intersect, if they are not parallel.
   (3) If Oscar graduates from college, he will either look for employment or go to graduate school.
   (4) If Virginia graduates from college and goes to graduate school, then she will not major in mathematics.
   (5) If Virginia graduates from college and has an A average in mathematics, she will receive a graduate fellowship.
   (6) Passing the algebra course is a necessary condition for Bill to graduate.
   (7) A sufficient condition for a triangle to satisfy the Pythagorean Theorem is that it is a right triangle.
   (8) A necessary condition for two triangles to be similar is that they have equal sides.
   (9) A triangle is equilateral only if its three angles are equal or its three sides are equal.
   (10) Three points lie on a circle only if they are not collinear.

D. Give a truth table for each of the following formulas. (Answer for 3.)
   (1) $A \rightarrow (B \rightarrow \neg A)$
   (2) $(A \wedge B \rightarrow C) \leftrightarrow (A \rightarrow C) \vee (B \rightarrow C)$
   (3) $A \vee \neg B \rightarrow C \wedge \neg A$
   (4) $A \wedge \neg A \rightarrow B \vee \neg C$
   (5) $(A \wedge \neg B \wedge C \wedge D) \vee (A \wedge B \wedge \neg C \wedge \neg D)$

E. Show that each of the following pairs of formulas have the same truth table:
   (1) $A \vee (A \wedge B)$, A
   (2) $A \wedge (A \vee B)$, A
   (3) $A \leftrightarrow B$, $(A \wedge B) \vee (\neg A \wedge \neg B)$
   (4) $\neg(A \wedge B)$, $\neg A \vee \neg B$
   (5) $A \rightarrow B$, $\neg A \vee B$
   (6) $\neg(A \rightarrow B)$, $A \wedge \neg B$
   (7) $A \rightarrow (B \rightarrow C)$, $A \wedge B \rightarrow C$
   (8) $A \vee B \rightarrow C$, $(A \rightarrow C) \wedge (B \rightarrow C)$

F. Construct a tree starting from sentential variables for each of the following formulas:
   (1) $A \rightarrow (B \rightarrow \neg A)$
   (2) $\neg A \wedge B \leftrightarrow (B \rightarrow C \vee A)$
   (3) $A \vee \neg B \rightarrow (\neg(C \wedge B) \leftrightarrow \neg A \wedge \neg B)$
   (4) $A \wedge (B \vee \neg C) \rightarrow A \wedge (D \rightarrow C)$
   (5) $\neg(A \wedge (B \leftrightarrow C)) \vee D \rightarrow C \wedge \neg D$