Chapter 2

Arguments, Valid and Invalid

In this chapter, we describe the important notion of a *proof* in the Sentential Calculus, but first we define some related concepts.

2.1 TAUTOLOGIES

Each formula in the Sentential Calculus has a truth table. If the last column of the truth table has all "T's," then that formula is called a *tautology*. A formula is a *tautology* if it is true for all possible truth values of its sentential variables. The following formulas are examples of tautologies: $A \vee \neg A$, $A \wedge \neg A \rightarrow (B \vee C \rightarrow D)$, $A \wedge B \leftrightarrow A \wedge B$, and $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$. If a formula is false for all truth values of its sentential variables, it is called a *contradiction*. The negation of a tautology will always be a contradiction and vice versa. The formulas $A \wedge \neg A$ and $A \leftrightarrow \neg A$ are contradictions that frequently occur in practice.

If P and Q are formulas such that $P \rightarrow Q$ is a tautology, we say P *tautologically implies* Q, which is expressed symbolically as $P \Rightarrow Q$. If $P \leftrightarrow Q$ is a tautology, we say P is *tautologically equivalent* to Q, symbolically $P \Leftrightarrow Q$. For example, $A \Leftrightarrow A \wedge A$ because $A \leftrightarrow A \wedge A$ is a tautology, and $A \Rightarrow A \vee B$ because $A \rightarrow A \vee B$ is a tautology.

Usually, if we have a disjunction of formulas or a conjunction of formulas, we omit parentheses because the formulas are tautologically equivalent. So, for example, we normally will not distinguish between the two tautologically equivalent formulas $A \wedge (B \wedge C)$ and $(A \wedge B) \wedge C$, and we will write each as $A \wedge B \wedge C$.

If $P_1, P_2, P_3, \ldots, P_n$ and Q are formulas, we say $P_1, P_2, P_3, \ldots, P_n$ *tautologically imply* Q if $P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n \rightarrow Q$ is a tautology. As a consequence, we have the following theorem:

Theorem 2.1 *$P_1, P_2, P_3, \ldots, P_n$ tautologically imply Q if and only if whenever $P_1, P_2, P_3, \ldots, P_n$ are all true, Q is true also.*

Proof: Suppose $P_1, P_2, P_3, \ldots, P_n$ tautologically imply Q. Then $P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n \rightarrow Q$ is a tautology. If $P_1, P_2, P_3, \ldots$ and P_n are all true, to make the conditional statement true, Q must be true.

Conversely, suppose Q is true whenever $P_1, P_2, P_3, \ldots,$ and P_n are all true. Then $P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n \rightarrow Q$ is a tautology because if all the P_i's are true, by hypothesis, Q is true, so the conditional is of the form $T \rightarrow T$, which is true. If some P_i is false, then the conditional statement is of the form $F \rightarrow T$ or $F \rightarrow F$, depending on whether Q is true or false. In both cases, the conditional is true. Thus, the conditional statement is always true, therefore, it is a tautology. ■

Theorem 2.1 allows us to shorten the time it takes to determine whether a conditional statement is a tautology. For example, to determine whether the formulas $A \rightarrow B$, $\neg A \rightarrow C$, and $C \rightarrow B$ tautologically imply B, we prepare a truth table with eight lines. The only lines we have to look at, however, are those in which the three formulas $A \rightarrow B$, $\neg A \rightarrow C$, and $C \rightarrow B$ are all true.

A	B	C	$A \rightarrow B$	$\neg A \rightarrow C$	$C \rightarrow B$	B
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F			
T	F	F	F			
F	T	T	T	T	T	T
F	T	F	T	F		
F	F	T	T	T	F	
F	F	F	T	F		

Lines 1, 2, and 5 are the only lines of the truth table where the three formulas are all true; on these lines, B is also true. Therefore, the three formulas $A \rightarrow B$, $\neg A \rightarrow C$, and $C \rightarrow B$ tautologically imply B.

On the other hand, suppose we want to show that the three formulas $A \rightarrow B$, $C \rightarrow \neg A$, and $C \rightarrow B$ do *not* tautologically imply B. All we have to do is find truth values for A, B, and C so that the three formulas are true but B is false. Starting with the conclusion, B, being false, we see that the first premise, $A \rightarrow B$, has a false consequent. In order for that to be true, A must be false. The second premise, $C \rightarrow \neg A$, now has a true consequent, so that is true whether C is true or false. The third premise, $C \rightarrow B$, however, has a false consequent, so C must be false to make it true. If we just construct the one line of the truth table

A	B	C	$A \to B$	$C \to \neg A$	$C \to B$	B
F	F	F	T	T	T	F

we see that it is possible to assign truth values to A, B, and C, so that the three formulas are true, but B is false. Thus, the three formulas do not tautologically imply B.

This discussion is leading up to the notion of a *valid argument*. An *argument* is a set of premises, $P_1, P_2, P_3, \ldots, P_n$, and a conclusion Q. An argument is *valid* if and only if whenever the premises are true, the conclusion is also true. That is,

$$P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n \Rightarrow Q \text{ or}$$
$$P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n \to Q \text{ is a tautology, or}$$
$$P_1, P_2, P_3, \ldots, \text{and } P_n \text{ tautologically imply } Q.$$

Example 2.1

Determine the validity of the following argument: If the dog is barking, then the dog is not in the house. If the dog is in the house, then someone is at the front door if the dog is barking. A necessary condition for the dog to be in the house is that the dog is barking. Therefore, someone is not at the front door unless the dog is barking.

Solution: Let the following letters represent the atomic sentences:

B: The dog is barking.
H: The dog is in the house.
D: Someone is at the front door.

Translating the argument into logical notation, we obtain the following:

Premises: $B \to \neg H$, $H \to (B \to D)$, $H \to B$
Conclusion: $\neg B \to \neg D$ (or $\neg D \vee B$)

We shall try to assign truth values to the atomic sentences so that the premises are true and the conclusion false. In order for $\neg B \to \neg D$ to be false, B must be false and D true. (A conditional statement is false if and only if the antecedent is true and the consequent false.) Moreover, we see that if H is false, all three premises are true. (The first premise is of the form $F \to T$; the second, $F \to (F \to T)$; and the third, $F \to F$.) Thus, the one line of the truth table

B	D	H	$B \to \neg H$	$H \to (B \to D)$	$H \to B$	$\neg B \to \neg D$
F	T	F	T	T	T	F

shows that it is possible to assign truth values to the atomic sentences in such a way that the premises are true but the conclusion is false. Such an assignment is sometimes called a *counter-example*. Consequently, the argument is *not* valid.

Example 2.2

Determine the validity of the following argument: If the dog is barking, then the dog is in the house. If the dog is in the house, then someone is at the front door unless the dog is not barking. Indeed, the dog is barking. Therefore, someone is at the front door.

Solution: First we translate the argument into logical notation. Using the same letters for atomic sentences as in Example 2.1, we obtain the following:

Premises: $B \to H$, $H \to (B \to D)$, B

Conclusion: D

If we start, as we did in the preceding example, to try to show that the argument is invalid, we would soon come to an impasse. (Try it!) Thus, we construct the whole truth table:

B	D	H	$B \to H$	$H \to (B \to D)$	B	D
T	T	T	T	T	T	T
T	T	F	F			
T	F	T	T	F		
T	F	F	F			
F	T	T	T	T	F	
F	T	F	T	T	F	
F	F	T	T	T	F	
F	F	F	T	T	F	

The only line of the truth table in which all the premises are true is the first line, and on this line, the conclusion is also true. Therefore, the argument is valid.

It is important to note that *the validity of an argument depends only on its structure*. The validity or nonvalidity of the arguments in Examples 2.1 and 2.2 did *not* depend on the interpretation or meaning of B, H, and D. It only depended on the *structure* of the premises and the conclusion.

For more complicated arguments, the truth table method to prove validity gets rather cumbersome. If there were ten atomic sentences, for example, the truth table would have $2^{10} = 1024$ lines. Although the problem can be done on a computer, it is still advantageous to have another method to show that an argument is valid. (The counter-example method, as illustrated in Example 2.1, is usually the best method to show that an argument is *not* valid.)

The method we shall describe to show that an argument is valid is called a *proof*, a *deduction*, or an *inference*. We shall give a few general rules, called *rules of inference*, that are sufficient to construct a "proof" of any valid argument in the Sentential Calculus. A rule of inference is a procedure for deducing a consequence from premises. The rules must be *valid* or *preserve truth*; that is, whenever the premises are true, the consequence must be true. Otherwise, there is no guarantee that the argument is valid.

2.2 RULE FOR TAUTOLOGIES

The first such rule we shall discuss is called the Rule for Tautologies, or Rule *T*.

Rule *T* (Rule for Tautologies) *A formula Q is deducible from formulas* $P_1, P_2, \ldots, P_n$ *if* $P_1 \wedge P_2 \wedge \cdots \wedge P_n \Rightarrow Q$. *(That is,* $P_1 \wedge P_2 \wedge \cdots \wedge P_n \rightarrow Q$ *is a tautology.) In particular, if Q itself is a tautology, then Q is deducible from the empty set of premises.*

It follows from Theorem 2.1 that Rule *T* is valid. Rules 1 through 4 listed below are four examples of Rule *T*:

Rule 1. From P and $P \rightarrow Q$ deduce Q.

Rule 2. From $\neg Q$ and $P \rightarrow Q$ deduce $\neg P$.

Rule 3. From $\neg P$ and $P \vee Q$ deduce Q.

Rule 4. From $\neg(\neg P)$ deduce P and from P deduce $\neg(\neg P)$.

Our preliminary definition of a *proof* is as follows:

By a *(formal) proof of Q from the premises* $P_1, P_2, \ldots, P_n$, we mean a finite sequence of formulas, $R_1, R_2, \ldots, R_m$, where R_m is Q, and each R_i is either one of the premises P_j or is deducible from preceding R_k's in the sequence using a valid rule of inference.

We use the notation "$P_1, P_2, \ldots, P_n \vdash Q$" if there is a proof of Q from $P_1, P_2, \ldots, P_n$. The symbol "$\vdash$" is sometimes called a "turnstile."

As an illustration of a formal proof, we shall prove that the following argument is valid.

Premises: $A \rightarrow \neg B,\ C \vee B,\ D \rightarrow \neg C,\ D$

Conclusion: $\neg A$

First, we shall describe informally how to deduce $\neg A$ from the premises:

Step 1: From D and $D \rightarrow \neg C$ deduce $\neg C$ using Rule 1.

Step 2: From $\neg C$ and $C \vee B$ deduce B using Rule 3.

Step 3: From B deduce $\neg(\neg B)$ using Rule 4.

Step 4: From $\neg(\neg B)$ and $A \rightarrow \neg B$ deduce $\neg A$ using Rule 2.

Somewhat more formally, the proof could be written as follows:

Proof	Reason
1. $A \to \neg B$	Premise
2. $C \vee B$	Premise
3. $D \to \neg C$	Premise
4. D	Premise
5. $\neg C$	T (Rule 1)
6. B	T (Rule 3)
7. $\neg(\neg B)$	T (Rule 4)
8. $\neg A$	T (Rule 2)

The proof is the finite sequence of formulas given on lines 1 through 8 above. Each formula in the proof is either a premise (1 through 4) or is derivable from preceding formulas in the sequence using a valid rule of inference (5 through 8). That is, line 5 is derived from lines 3 and 4 using Rule 1, line 6 is derived from lines 2 and 5 using Rule 3, and so forth. Normally, in a proof, we write the given premises at the beginning of the proof, but this is not required, because a premise can be written on any line of a proof.

2.3 RULE FOR PREMISES

Formally, our rule for premises, or Rule P, is as follows:

Rule P (Rule for Premises) *A premise may occur on any line of a proof.*

Moreover, in each proof, we shall indicate which premises each line depends upon. To do this, we shall associate with each premise a number. We shall use the number of the line where the premise first occurs. (Any other consistent numbering system would be just as good.) Thus, our formal proof of the preceding argument will look like this:

Premise Numbers	Line Numbers	Proof	Reasons
{1}	1.	$A \to \neg B$	P
{2}	2.	$C \vee B$	P
{3}	3.	$D \to \neg C$	P
{4}	4.	D	P
{3, 4}	5.	$\neg C$	3, 4 T
{2, 3, 4}	6.	B	2, 5 T
{2, 3, 4}	7.	$\neg(\neg B)$	6 T
{1, 2, 3, 4}	8.	$\neg A$	1, 7 T

Lines 1 through 4 are self-evident. To derive line 5, we used lines 3 and 4 and Rule T (Rule 1); thus, line 5 depends upon premises 3 and 4. To derive line 6, we used lines 2 and 5 and Rule T (Rule 3). Thus, line 6 depends upon premises 2, 3, and 4. Similarly, for the remaining part of the proof. In general, numbers preceding the line number in a proof are the numbers of the premises on which that line depends.

To construct proofs when the arguments are more complicated, we shall have to become more proficient in the use of Rule T, which means familiarizing ourselves with some more tautologies. For that purpose, we have made a list of some useful tautologies at the end of this chapter (see Section 2.7). In fact, this list of tautologies is *complete*; that is, any valid argument can be proved using tautologies from this list. Actually a much smaller list would be adequate. We shall delay the discussion of this type of completeness to Chapter 16 (see Section 16.3). In this list of tautologies, we have included a name for each tautology for historical reasons or, in some cases, as a mnemonic device. In most cases, there is no good reason for memorizing these names.

Clearly, every valid argument has a very simple proof. If $P_1, P_2, \ldots, P_n$ are the premises and Q is the conclusion of the valid argument, then

$$P_1 \wedge P_2 \wedge \cdots \wedge P_n \to Q$$

is a tautology. Therefore, the first n lines of a proof are the n premises and the $n + 1$st line is Q. That is not the point, however. The point is to proceed with small, obvious steps. We illustrate this in the next example.

Example 2.3 Show that the following argument is valid by giving a formal proof.

Premises: $A \to B \vee C,\ A \to \neg B,\ C \to \neg D$

Conclusion: $A \to \neg D$

Solution: First, we'll give an informal proof. (The numbers in parentheses refer to the tautologies given in Section 2.7. It will be useful for the reader to have a copy of this list.) We'll denote the premises by P_1: $(A \to B \vee C)$; P_2: $(A \to \neg B)$; and P_3: $(C \to \neg D)$.

Step 1: $P_2 \vdash B \to \neg A$ (20).

Step 2: $\vdash \neg A \to \neg A \vee \neg D$ (5).

Step 3: $\vdash \neg D \to \neg A \vee \neg D$ (5, 12).

Step 4: $B \to \neg A,\ \neg A \to \neg A \vee \neg D \vdash B \to \neg A \vee \neg D$ (7).

Step 5: $P_3,\ \neg D \to \neg A \vee \neg D \vdash C \to \neg A \vee \neg D$ (7).

Step 6: $B \to \neg A \vee \neg D,\ C \to \neg A \vee \neg D \vdash B \vee C \to \neg A \vee \neg D$ (8).

Step 7: $P_1,\ B \vee C \to \neg A \vee \neg D \vdash A \to \neg A \vee \neg D$ (7).

Step 8: $\vdash (\neg A \vee \neg D) \to (A \to \neg D)$ (17).

Step 9: $A \rightarrow \neg A \vee \neg D, \neg A \vee \neg D \rightarrow (A \rightarrow \neg D) \vdash A \rightarrow (A \rightarrow \neg D)$ (7).
Step 10: $A \rightarrow (A \rightarrow \neg D) \vdash A \wedge A \rightarrow \neg D$ (21).
Step 11: $\vdash A \rightarrow A \wedge A$ (19).
Step 12: $A \rightarrow A \wedge A, A \wedge A \rightarrow \neg D \vdash A \rightarrow \neg D$ (7).

(Later we shall introduce another rule, and give an easier proof of this argument.) The formal proof is as follows:

{1}	1.	$A \rightarrow B \vee C$	P
{2}	2.	$A \rightarrow \neg B$	P
{3}	3.	$C \rightarrow \neg D$	P
{2}	4.	$B \rightarrow \neg A$	2 T
	5.	$\neg A \rightarrow \neg A \vee \neg D$	T
	6.	$\neg D \rightarrow \neg A \vee \neg D$	T
{2}	7.	$B \rightarrow \neg A \vee \neg D$	4, 5 T
{3}	8.	$C \rightarrow \neg A \vee \neg D$	3, 6 T
{2, 3}	9.	$B \vee C \rightarrow \neg A \vee \neg D$	7, 8 T
{1, 2, 3}	10.	$A \rightarrow \neg A \vee \neg D$	1, 9 T
	11.	$\neg A \vee \neg D \rightarrow (A \rightarrow \neg D)$	T
{1, 2, 3}	12.	$A \rightarrow (A \rightarrow \neg D)$	10, 11 T
{1, 2, 3}	13.	$A \wedge A \rightarrow \neg D$	12 T
	14.	$A \rightarrow A \wedge A$	T
{1, 2, 3}	15.	$A \rightarrow \neg D$	13, 14 T

Notice that the formulas on lines 5, 6, 11, and 14 are tautologies, so no premise numbers are given. Frequently, such lines are omitted.

We are implicitly using a rule of substitution when we use the list of tautologies given at the end of the chapter. Each of the letters "P," "Q," and "R" represent a formula. So, for example, the valid rule

$$\neg(A \wedge B), (A \wedge B) \vee (B \rightarrow C) \vdash B \rightarrow C$$

is obtained from Modus Tollendo Ponens (3) by substituting "$A \wedge B$" for each occurrence of "P" and "$B \rightarrow C$" for each occurrence of "Q." Similarly, the following formula is a tautology:

$$(A \wedge \neg B \rightarrow C) \vee \neg(A \wedge \neg B \rightarrow C).$$

It is a form of the Rule of the Excluded Middle (10) with "$A \wedge \neg B \rightarrow C$" substituted for "$P$."

In the next example, we shall omit the numbers of the tautologies. (The reader should fill in the numbers for himself or herself.)

Example 2.4.

Show that the following argument is valid by giving a formal proof:

Premises: $A \rightarrow B$, $\neg A \rightarrow \neg C \vee D$, $C \wedge \neg D$

Conclusion: B

Solution: The informal proof is as follows:

Step 1: $P_3 \vdash \neg(\neg C \vee D)$.

Step 2: $\neg(\neg C \vee D), P_2 \vdash A$.

(To obtain step 2, we used two rules. What are they?)

Step 3: $A, P_1 \vdash B$.

The formal proof is as follows:

{1}	1.	$A \rightarrow B$	P
{2}	2.	$\neg A \rightarrow \neg C \vee D$	P
{3}	3.	$C \wedge \neg D$	P
{3}	4.	$\neg(\neg C \vee D)$	3 T
{2, 3}	5.	A	2, 4 T
{1, 2, 3}	6.	B	1, 5 T

WARNING! Beginning logic students frequently make two common errors:

1. The fallacy of asserting the consequent: From Q and $P \rightarrow Q$, deduce P.
2. The fallacy of denying the antecedent: From $\neg P$ and $P \rightarrow Q$, deduce $\neg Q$.

Neither of these rules is valid, as is easily seen from truth tables. There are disastrous consequences from using invalid rules. A contradiction is deducible from an invalid rule and, by the Rule of Absurdity (9), every formula is deducible from a contradiction.

We shall show that we can derive a contradiction, $A \wedge \neg A$, using the fallacy of asserting the consequent, *FAC*:

1.	$A \wedge \neg A \rightarrow A \vee \neg A$	T
2.	$A \vee \neg A$	T
3.	$A \wedge \neg A$	1, 2 *FAC*

In *FAC*, we substituted "$A \wedge \neg A$" for "P" and "$A \vee \neg A$" for "Q." In general, if the rule

$$\text{from } P_1, P_2, \ldots, P_n \text{ deduce } Q$$

is invalid, then there is an assignment, s, of truth values for the sentential variables in the rule so that $P_1, P_2, \ldots, P_n$ are each true and Q is false. For each sentential variable A that occurs in the rule

$$\text{if } s(A) = T, \text{ replace } A \text{ by } B \vee \neg B,$$
$$\text{if } s(A) = F, \text{ replace } A \text{ by } B \wedge \neg B,$$

where B is a sentential variable that does not occur in the rule. Then, each of $P_1, P_2, \ldots, P_n$ becomes a tautology, and Q becomes a contradiction. Thus, a contradiction is deducible using the invalid rule.

If a contradiction is deducible in a system, the system is called *inconsistent*. Inconsistent systems are trivial and uninteresting because, in such a system, every formula is provable. It is, however, of no small interest to identify which systems are inconsistent, and a good deal of effort is spent on just such activity in more complicated settings.

2.4 RULE OF CONDITIONAL PROOF

Although the two rules of inference that we have, Rules P and T, are sufficient to prove any valid argument, it is convenient to introduce two additional rules in order to simplify proofs (see Theorems 16.3 and 16.4). The first is called the Rule of Conditional Proof, or Rule CP.

Rule *CP* (Rule of Conditional Proof)

If Q is deducible from P and a set of premises Γ then $P \to Q$ is deducible from Γ alone. In symbols, if $\Gamma, P \vdash Q$, then $\Gamma \vdash P \to Q$.

We shall first illustrate how Rule CP can be used to shorten the proof of the argument given in Example 2.3. We use Rule CP when we want to prove a conditional, $P \to Q$. The method is to take P as an additional premise, then deduce Q.

{1}	1.	$A \to B \vee C$	P
{2}	2.	$A \to \neg B$	P
{3}	3.	$C \to \neg D$	P
{4}	4.	A	P(for CP)
{1, 4}	5.	$B \vee C$	1, 4 T
{2, 4}	6.	$\neg B$	2, 4 T
{1, 2, 4}	7.	C	5, 6 T
{1, 2, 3, 4}	8.	$\neg D$	3, 7 T
{1, 2, 3}	9.	$A \to \neg D$	4, 8 CP

On line 4, we take A as an additional premise for the Rule CP. On lines 5 through 8, we deduce $\neg D$. At this point, we apply the Rule CP, so on line 9, we say that the conditional $A \to \neg D$ is deducible from our original set of premises. The numerals 4 and 8 on line 9 are the numbers of the line on which the temporary premise appeared, 4, and on which the conclusion appeared, 8. Under certain conditions, then, the Rule *CP* allows us to drop a premise. This is an important aspect of the rule. *The last line of a proof cannot depend upon any premises that were not in the original set of premises.* Otherwise, we're changing the question. It is trivial to prove any formula P if P is taken as an additional premise.

Let us examine the preceding derivation a little more closely. According to our preliminary definition of a proof, this is *not* a proof because, on line 4, we used a premise that was not in the original set of premises. On the last line, however, we do have the conclusion, and it only depends upon premises from the original set. We shall now give the "real" definition of a proof, to validate this type of phenomenon. It should be all right to introduce premises on any line of a proof as long as the last line only depends upon premises from the original set.

Definition 2.1 A *formal proof of Q from the premises $P_1, P_2, \ldots, P_n$* is a finite sequence of ordered pairs $\langle \alpha_1, R_1 \rangle, \langle \alpha_2, R_2 \rangle, \ldots, \langle \alpha_m, R_m \rangle$ where each α_i is a set of formulas and each R_i is a formula such that

1. $\alpha_m \subseteq \{P_1, P_2, \ldots, P_n\}$ and $R_m = Q$.
2. Each R_i is either a premise or is deducible from preceding R_j's in the sequence using a valid rule of inference.
3. If R_i is a premise then $\alpha_i = \{R_i\}$.
4. If R_i is deducible from preceding R_j's in the sequence using a valid rule of inference then α_i is defined by the rule.

In Chapter 12 (see Section 12.2), we state each rule in such a way that its effect on the α_i's is clear. In this section, we shall explain the effect less formally.

In our proofs we abbreviate the α_i's by replacing each premise by the line number where the premise occurs. The symbol "$P_1, P_2, \ldots, P_n \vdash Q$" is an abbreviation for "there is a proof of Q from $P_1, P_2, \ldots, P_n$." Similarly, if Γ is a set of formulas and Q is a formula, "$\Gamma \vdash Q$" is an abbreviation for "there is a proof of Q from Γ."

Part 1 of Definition 2.1 states that the conclusion depends only on premises from the original set of premises; part 2 contains our preliminary definition of a proof; and parts 3 and 4 explain how to construct the set of premise numbers for each line of a proof. Part 3 says that if the formula on line i is a premise, then the set of premise numbers for line i is $\{i\}$. Thus, each α_i is the set of all formulas on which formula R_i depends. Auxiliary premises may be used in the course of a proof, but the conclusion must only depend on premises from the original set.

To illustrate part 4 of Definition 2.1, if the rule is Rule T and the tautology is $R_j \wedge R_k \to R_i$, then $\alpha_i = \alpha_j \cup \alpha_k$. See lines 5, 6, and 7 of the preceding proof, for example. Line 7 is obtained from lines 5 and 6 by using a tautology. Thus, the set of premise numbers for line 7 is {1, 2, 4} and {1, 2, 4} = {1, 4} ∪ {2, 4}, which is the union of the sets of premise numbers for lines 5 and 6.

On the other hand, if the rule is *CP* and R_j is the premise for *CP*, then $\alpha_j = \{j\}$. If we then deduce R_k ($\alpha_k = \Gamma \cup \alpha_j$) and if we use *CP* to get the next line, then R_{k+1} is $R_j \rightarrow R_k$ and $\alpha_{k+1} = \alpha_k - \{j\}$. On line 9 of the preceding proof, for example, we used Rule *CP*. The premise for Rule *CP*, *A*, was on line 4. We derived $\neg D$ on line 8. The set of premise numbers on line 8 is {1, 2, 3, 4}; thus, the formula on line 9 is $A \rightarrow \neg D$, and the set of premise numbers is {1, 2, 3, 4} − {4} = {1, 2, 3}.

Next, we shall show Rule *CP* is a valid rule. (Otherwise, we could derive a contradiction.)

Theorem 2.2 *Rule CP is valid.*

Proof: Suppose $P_1, P_2, \ldots, P_n$ are the premises in Γ. To show Rule *CP* is valid, we must show the following: If

(1) $P_1 \wedge P_2 \wedge \cdots \wedge P_n \wedge P \Rightarrow Q$,

then

(2) $P_1 \wedge P_2 \wedge \cdots \wedge P_n \Rightarrow (P \rightarrow Q)$.

Suppose that (1) is true; that is,

(3) $P_1 \wedge P_2 \wedge \cdots \wedge P_n \wedge P \rightarrow Q$

is a tautology. Then we must show (2) is true; that is,

(4) $P_1 \wedge P_2 \wedge \cdots \wedge P_n \rightarrow (P \rightarrow Q)$

is a tautology. However, (3) ↔ (4) is a tautology (the Rule of Exportation-Importation, (21), with "$P_1 \wedge P_2 \wedge \cdots \wedge P_n$" substituted for "*P*," "*P*" substituted for "*Q*," and "*Q*" substituted for "*R*"). Thus, if (3) is a tautology, so is (4), so (1) implies (2). ■

2.5 RULE OF INDIRECT PROOF

The last rule we shall consider for the Sentential Calculus is the Rule of Indirect Proof, or Rule *IP*.

Rule *IP* (Rule of Indirect Proof) *If any contradiction, $Q \wedge \neg Q$, where Q is any formula, is deducible from $\neg P$ and a set of premises Γ, then P is deducible from Γ alone.* In symbols, if $\neg P, \Gamma \vdash Q \wedge \neg Q$, then $\Gamma \vdash P$.

Theorem 2.3 *Rule IP is valid.*

Proof: If $P_1, P_2, \ldots, P_n$ are the premises in Γ, we must show that

(1) $P_1 \wedge P_2 \wedge \cdots \wedge P_n \wedge \neg P \Rightarrow Q \wedge \neg Q$

implies

(2) $P_1 \wedge P_2 \wedge \cdots \wedge P_n \Rightarrow P$.

Suppose (1) is true. Then

(3) $P_1 \wedge P_2 \wedge \cdots \wedge P_n \wedge \neg P \rightarrow Q \wedge \neg Q$

is a tautology. The consequent, $Q \wedge \neg Q$, of (3) is always false. Thus, for (3) to be a tautology, one of the conjuncts in the antecedent must be false. If any of the P_i's are false, then

(4) $P_1 \wedge P_2 \wedge \cdots \wedge P_n \rightarrow P$

is true (because the antecedent is false). If $\neg P$ is false, then P is true. This also implies that (4) is true (because the consequent is true). Therefore, (4) is a tautology, so (2) is true. ■

Example 2.5 Show that the following argument is valid by giving a formal proof.

Premises: $A \rightarrow (B \rightarrow C)$, $C \rightarrow \neg D$, $\neg E \rightarrow D$, $A \wedge B$

Conclusion: E

Solution: We shall give an indirect proof and take $\neg E$ as an additional premise. The informal proof is as follows:

Step 1: $\neg E, P_3 \vdash D$.

Step 2: $D, P_2 \vdash \neg C$.

Step 3: $P_4 \vdash A$.

Step 4: $A, P_1 \vdash B \rightarrow C$.

Step 5: $P_4 \vdash B$.

Step 6: $B, B \rightarrow C \vdash C$.

Step 7: $C, \neg C \vdash C \wedge \neg C$.

Thus, it follows from the Rule of Indirect Proof that $P_1, P_2, P_3, P_4 \vdash E$. The formal proof is as follows:

{1}	1.	$A \rightarrow (B \rightarrow C)$	P
{2}	2.	$C \rightarrow \neg D$	P
{3}	3.	$\neg E \rightarrow D$	P
{4}	4.	$A \wedge B$	P

{5}	5.	$\neg E$	P(for IP)
{3, 5}	6.	D	3, 5 T
{2, 3, 5}	7.	$\neg C$	2, 6 T
{4}	8.	A	4 T
{1, 4}	9.	$B \rightarrow C$	1, 8 T
{4}	10.	B	4 T
{1, 4}	11.	C	9, 10 T
{1, 2, 3, 4, 5}	12.	$C \wedge \neg C$	7, 11 T
{1, 2, 3, 4}	13.	E	5, 12 IP

On line 12, we derived a contradiction using the four original premises and the premise given on line 5, $\neg E$. Thus, it follows from the Rule of Indirect Proof that E is deducible from the original four premises, and that is how we derived line 13 of the proof. Similar to the Rule *CP*, the Rule *IP* allows us to omit a temporary premise. The line numbers on line 13 are 5, the line number where the temporary premise occurred, and 12, the line number where the contradiction occurred.

In the following section, we give some guidelines for proofs in the Sentential Calculus and then give some more examples. The guidelines are possible strategies to try in a proof. Proofs are not always straightforward procedures.

2.6 GUIDELINES

1. To prove a conjunction, $P \wedge Q$, prove each of the conjuncts, and then use the valid rule, $P, Q \vdash P \wedge Q$.
2. To prove a conditional, $P \rightarrow Q$, take P as a premise, derive Q, and then use the Rule *CP*.
3. To prove a disjunction, $P \vee Q$,
   (a) Prove either P or Q, then use the rule $P \vdash P \vee Q$ or $Q \vdash P \vee Q$.
   or
   (b) Take $\neg P$ as a premise, derive Q, then use Rule *CP* to deduce $\neg P \rightarrow Q$. Then use the rule $\neg P \rightarrow Q \vdash P \vee Q$.
4. To prove a biconditional, $P \leftrightarrow Q$, prove $P \rightarrow Q$ and $Q \rightarrow P$. Then use the rule $P \rightarrow Q, Q \rightarrow P \vdash P \leftrightarrow Q$.

   If none of the preceding methods work, try
5. An indirect proof: To prove P, take $\neg P$ as a premise and derive a contradiction.
6. An alternative proof:
   (a) To prove P, find some sentence Q such that $Q \rightarrow P$ and $\neg Q \rightarrow P$. Then use the rule $Q \rightarrow P, \neg Q \rightarrow P \vdash P$. (See 8 on the list of tautologies.)

(b) To prove R from the premise $P \vee Q$, prove $P \to R$ and $Q \to R$, and then use tautology 8 to obtain $P \vee Q \to R$.

First, we'll give examples of alternative proofs.

Example 2.6

Show that the following arguments are valid by giving a formal proof:

(a) **Premises:** $A \to B$, $\neg A \to C$, $C \to \neg D$, $\neg B \to D$
Conclusion: B

(b) **Premises:** $A \vee B$, $C \to \neg A$, $\neg B \vee D$, $D \to \neg C$
Conclusion: $\neg C$

Solution:

(a)

{1}	1.	$A \to B$	P
{2}	2.	$\neg A \to C$	P
{3}	3.	$C \to \neg D$	P
{4}	4.	$\neg B \to D$	P

(We shall use lines 2, 3, and 4 to derive $\neg A \to B$, and then, with line 1, use an alternative proof to derive B.)

{2, 3}	5.	$\neg A \to \neg D$	2, 3 T
{4}	6.	$\neg D \to B$	4 T
{2, 3, 4}	7.	$\neg A \to B$	5, 6 T
{1, 2, 3, 4}	8.	B	1, 7 T

On line 8, we used an alternative proof.

(b)

{1}	1.	$A \vee B$	P
{2}	2.	$C \to \neg A$	P
{3}	3.	$\neg B \vee D$	P
{4}	4.	$D \to \neg C$	P

We shall prove $A \to \neg C$ and $B \to \neg C$.

{2}	5.	$A \to \neg C$	2 T
{6}	6.	B	P (for CP)
{3, 6}	7.	D	3, 6 T
{3, 4, 6}	8.	$\neg C$	4, 7 T
{3, 4}	9.	$B \to \neg C$	6, 8 CP
{2, 3, 4}	10.	$A \vee B \to \neg C$	5, 9 T
{1, 2, 3, 4}	11.	$\neg C$	1, 10 T

We obtained line 10 by using tautology 8.

There are other ways of proving parts (a) and (b) in Example 2.6, but an alternative proof is nonetheless often useful.

Example 2.7

For each of the following arguments, give a formal proof if it is valid; if not, give a counter-example—that is, assign truth values to the atomic sentences so that the premises are true and the conclusion is false.

(a) **Premises:** $A \rightarrow B \vee C$, $B \rightarrow \neg A$, $\neg D \rightarrow \neg C$
Conclusion: $\neg A \vee D$

(b) **Premises:** $A \rightarrow \neg B$, $(A \vee \neg B) \vee C$, B
Conclusion: $B \wedge C$

(c) **Premises:** $\neg A \rightarrow (B \vee C)$, $\neg B \rightarrow (\neg A \wedge D)$, $D \rightarrow (B \vee C)$
Conclusion: B

(d) **Premises:** $A \rightarrow C$, $\neg C \vee D$, $B \leftrightarrow D$, $B \rightarrow \neg(\neg A \wedge D)$
Conclusion: $A \leftrightarrow B$

Solution:

(a) We shall take A as a premise and try to derive D. (See 3(b) in the guidelines given above; $\neg A \vee D$ is tautologically equivalent to $A \rightarrow D$.) The following is an informal proof:

Step 1: $A, P_1 \vdash B \vee C$.
Step 2: $A, P_2 \vdash \neg B$.
Step 3: $B \vee C, \neg B \vdash C$.
Step 4: $C, P_3 \vdash D$.

Thus, by Rule *CP*, we obtain $A \rightarrow D$, which is tautologically equivalent to $\neg A \vee D$. The formal proof is as follows:

{1}	1.	$A \rightarrow B \vee C$	*P*
{2}	2.	$B \rightarrow \neg A$	*P*
{3}	3.	$\neg D \rightarrow \neg C$	*P*
{4}	4.	A	*P*(for *CP*)
{1, 4}	5.	$B \vee C$	1, 4 *T*
{2, 4}	6.	$\neg B$	2, 4 *T*
{1, 2, 4}	7.	C	5, 6 *T*
{1, 2, 3, 4}	8.	D	3, 7 *T*
{1, 2, 3}	9.	$A \rightarrow D$	4, 8 *CP*
{1, 2, 3}	10.	$\neg A \vee D$	9 *T*

(b) In this argument we are trying to prove a conjunction, $B \wedge C$. The method we shall use is described in Guideline 1, that is, deduce each of the conjuncts. One of the conjuncts is a premise, B, so that is easy to deduce. It remains to deduce C. For this, we shall use an indirect proof.

Informally, take $\neg C$ as a premise:

Step 1: $\neg C, P_2 \vdash A \vee \neg B$.

Step 2: $P_3, A \vee \neg B \vdash A$.

Step 3: $A, P_1 \vdash \neg B$.

Step 4: $\neg B, P_3 \vdash B \wedge \neg B$.

Thus, from Rule *IP*, we can deduce C.

Step 5: $C, P_3 \vdash B \wedge C$.

The following is a formal proof:

{1}	1.	$A \to \neg B$	P
{2}	2.	$(A \vee \neg B) \vee C$	P
{3}	3.	B	P
{4}	4.	$\neg C$	P(for *IP*)
{2, 4}	5.	$A \vee \neg B$	2, 4 T
{2, 3, 4}	6.	A	3, 5 T
{1, 2, 3, 4}	7.	$\neg B$	1, 6 T
{1, 2, 3, 4}	8.	$B \wedge \neg B$	3, 7 T
{1, 2, 3}	9.	C	4, 8 *IP*
{1, 2, 3}	10.	$B \wedge C$	3, 9 T

(c) To try to prove (c), we shall try an indirect proof; take $\neg B$ as a premise, and try to derive a contradiction.

Step 1: $\neg B, P_2 \vdash \neg A \wedge D$.

Step 2: $\neg A \wedge D \vdash \neg A$.

Step 3: $\neg A, P_1 \vdash B \vee C$.

Step 4: $\neg A \wedge D \vdash D$.

Step 5: $D, P_3 \vdash B \vee C$.

Step 6: $\neg B, B \vee C \vdash C$.

If C were false, we would have a contradiction, but we do not have any information to tell us that C is false. Let's try instead to find a counterexample.

Because B is the conclusion, B has to be false, and from the preceding discussion, C has to be true. Now we have to find truth assignments for A and D so that the premises are true. The following assignment works:

A	B	C	D	$\neg A \to B \vee C$	$\neg B \to (\neg A \wedge D)$	$D \to (B \vee C)$	B
F	F	T	T	T $(T \to T)$	T $(T \to T)$	T $(T \to T)$	F

Thus, if A and B are false, and C and D are true, the premises are true, and the conclusion is false, so the argument is not valid. Normally, to show an argument is not valid, try to prove it, and if you don't succeed, try to find a counter-example. One or the other must work.

(d) In this argument, we are trying to prove a biconditional, $A \leftrightarrow B$. The method we shall try is to deduce $A \to B$ and $B \to A$. (See Guideline 4.)

The formal proof is as follows:

{1}	1.	$A \to C$	P
{2}	2.	$\neg C \vee D$	P
{3}	3.	$B \leftrightarrow D$	P
{4}	4.	$B \to \neg(\neg A \wedge D)$	P
{5}	5.	A	P(for CP)
{1, 5}	6.	C	1, 5 T
{1, 2, 5}	7.	D	2, 6 T
{1, 2, 3, 5}	8.	B	3, 7 T
{1, 2, 3}	9.	$A \to B$	5, 8 CP
{10}	10.	B	P(for CP)
{4, 10}	11.	$\neg(\neg A \wedge D)$	4, 10 T
{4, 10}	12.	$A \vee \neg D$	11 T
{3, 10}	13.	D	3, 10 T
{3, 4, 10}	14.	A	12, 13 T
{3, 4}	15.	$B \to A$	10, 14 CP
{1, 2, 3, 4}	16.	$A \leftrightarrow B$	9, 15 T

Notice that to prove $A \to B$ (line 9), we used only premises 1, 2, and 3, and to prove $B \to A$ (line 15), we used only premises 3 and 4. To prove the biconditional, $A \leftrightarrow B$ (line 16), however, we used all four premises. Both in the assignments and in arguments occurring naturally, there may be redundant premises.

2.7 USEFUL TAUTOLOGIES

("P," "Q," and "R" denote arbitrary formulas.)

1. $P \wedge (P \to Q) \to Q$ (Rule of Detachment, Modus Ponens)
2. $\neg Q \wedge (P \to Q) \to \neg P$ (Modus Tollendo Tollens)
3. $\neg P \wedge (P \vee Q) \to Q$ (Modus Tollendo Ponens)
4. $P \wedge Q \to P$ (Rule of Simplification)
5. $P \to P \vee Q$ (Rule of Addition)
6. $\left.\begin{array}{l} P \to (Q \to P \wedge Q) \\ (P \to Q) \wedge (P \to R) \to (P \to Q \wedge R) \end{array}\right\}$ (Rule of Adjunction)

7. $(P \to Q) \land (Q \to R) \to (P \to R)$ (Rule of Hypothetical Syllogism)

8. $\left.\begin{array}{l} (P \to Q) \land (\neg P \to Q) \to Q \\ (P \lor Q \to R) \leftrightarrow (P \to R) \land (Q \to R) \end{array}\right\}$ (Rules of Alternative Proof)

9. $\left.\begin{array}{l} (P \to Q \land \neg Q) \to \neg P \\ P \land \neg P \to Q \end{array}\right\}$ (Rule of Absurdity)

10. $P \lor \neg P$ (Rule of the Excluded Middle)

11. $\neg(P \land \neg P)$ (Rule of Contradiction)

12. $\left.\begin{array}{l} P \lor Q \leftrightarrow Q \lor P \\ P \land Q \leftrightarrow Q \land P \end{array}\right\}$ (Commutative Rules)

13. $\left.\begin{array}{l} P \lor (Q \lor R) \leftrightarrow (P \lor Q) \lor R \\ P \land (Q \land R) \leftrightarrow (P \land Q) \land R \end{array}\right\}$ (Associative Rules)

14. $\left.\begin{array}{l} P \lor (Q \land R) \leftrightarrow (P \lor Q) \land (P \lor R) \\ P \land (Q \lor R) \leftrightarrow (P \land Q) \lor (P \land R) \end{array}\right\}$ (Distributive Rules)

15. $\left.\begin{array}{l} \neg(P \lor Q) \leftrightarrow \neg P \land \neg Q \\ \neg(P \land Q) \leftrightarrow \neg P \lor \neg Q \end{array}\right\}$ (De Morgan's Rules)

16. $\neg(\neg P) \leftrightarrow P$ (Rule of Double Negation)

17. $\left.\begin{array}{l} (P \to Q) \leftrightarrow \neg P \lor Q \\ \neg(P \to Q) \leftrightarrow P \land \neg Q \end{array}\right\}$ (Rules for the Conditional)

18. $\left.\begin{array}{l} (P \leftrightarrow Q) \leftrightarrow (P \to Q) \land (Q \to P) \\ (P \leftrightarrow Q) \leftrightarrow (P \land Q) \lor (\neg P \land \neg Q) \\ (P \leftrightarrow Q) \leftrightarrow (\neg P \lor Q) \land (P \lor \neg Q) \end{array}\right\}$ (Rules for the Biconditional)

19. $\left.\begin{array}{l} P \lor P \leftrightarrow P \\ P \land P \leftrightarrow P \end{array}\right\}$ (Idempotency Rules)

20. $\left.\begin{array}{l} (P \to Q) \leftrightarrow (\neg Q \to \neg P) \\ (P \to \neg Q) \leftrightarrow (Q \to \neg P) \\ (\neg P \to Q) \leftrightarrow (\neg Q \to P) \end{array}\right\}$ (Rules of Contraposition)

21. $(P \to (Q \to R)) \leftrightarrow (P \land Q \to R)$ (Rule of Exportation-Importation)

22. $\left.\begin{array}{l} P \lor (P \land Q) \leftrightarrow P \\ P \land (P \lor Q) \leftrightarrow P \end{array}\right\}$ (Rules of Absorption)

EXERCISES 2

A. For each of the following pairs of sentences, use truth tables to determine whether

(i) $(a) \Rightarrow (b)$,
(ii) $(b) \Rightarrow (a)$,
(iii) $(a) \Leftrightarrow (b)$,

(iv) none of these.

(Answers for 1, 7, 9, and 10.)

(1) **(a)** $A \to (B \to \neg C)$,	**(b)** $A \to (\neg B \to C)$
(2) **(a)** $A \wedge B \to \neg C$,	**(b)** $A \to (C \to \neg B)$
(3) **(a)** $(A \wedge B) \wedge \neg C$,	**(b)** $C \to (B \to A)$
(4) **(a)** A,	**(b)** $A \vee (B \wedge C)$
(5) **(a)** A,	**(b)** $(A \wedge B) \vee C$
(6) **(a)** $A \vee B \vee C$,	**(b)** $A \vee (\neg A \wedge B) \vee (\neg B \wedge C) \vee \neg C$
(7) **(a)** $(A \vee B) \wedge C$,	**(b)** $((A \wedge B) \vee (\neg A \wedge B) \vee (A \wedge \neg B)) \wedge C$
(8) **(a)** $(A \wedge B \wedge C) \vee (A \wedge B \wedge \neg C)$,	**(b)** $C \wedge \neg C$
(9) **(a)** $A \wedge B \wedge C$,	**(b)** $(A \vee \neg B) \wedge (B \vee \neg C) \wedge (C \vee \neg A)$
(10) **(a)** $(A \to B) \vee C$,	**(b)** $(A \leftrightarrow B) \wedge C$
(11) **(a)** $(A \to B) \to C$,	**(b)** $A \to (B \to C)$
(12) **(a)** $(A \leftrightarrow B) \leftrightarrow C$,	**(b)** $A \leftrightarrow (B \leftrightarrow C)$

B. Each of the following formulas is a tautology that is obtained by a substitution in one of the tautologies given in the list of useful tautologies. In each case, state which tautology it is and what the substitution is. (Answers for 1, 5, and 9.)

(1) $(A \vee B) \vee (C \wedge D) \leftrightarrow (A \vee B \vee C) \wedge (A \vee B \vee D)$
(2) $\neg A \wedge (\neg A \to B \vee C) \to B \vee C$
(3) $(A \wedge B) \vee (A \wedge \neg B \wedge C) \leftrightarrow A \wedge (B \vee (\neg B \wedge C))$
(4) $(A \wedge B) \vee (A \wedge B \wedge C) \leftrightarrow A \wedge B$
(5) $(A \wedge B \wedge C \to D \wedge \neg D) \to \neg(A \wedge B \wedge C)$
(6) $A \wedge \neg A \to A \vee \neg A$
(7) $(A \wedge \neg A) \vee \neg(A \wedge \neg A)$
(8) $(A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \leftrightarrow (A \wedge \neg B) \wedge (C \vee \neg C)$
(9) $\neg(A \vee \neg B \vee C) \leftrightarrow \neg A \wedge \neg(\neg B \vee C)$
(10) $(A \wedge B \to C \vee D) \leftrightarrow (A \to (B \to C \vee D))$

C. Translate each of the following arguments into logical notation using the given letters for atomic sentences. Also determine the validity of the argument using truth tables. (Answers for 2, 4, and 7.)

(1) If Oscar attends class, then either Miriam or Virginia attend class. Miriam is not attending class. Therefore, Virginia attends class if Oscar does. (O, M, V)

(2) If Oscar attends class, then so does Miriam, and if Miriam attends class, so does George. Oscar attends class unless George attends. Therefore, Miriam does not attend class. (O, M, G)

(3) If Oscar attends class, then Virginia attends class only if George attends. George does attend class. Therefore, if Oscar attends class, so does Virginia. (O, V, G)

(4) If both Oscar and George attend class, so does Virginia. George does attend class. Therefore, either Virginia attends class or Oscar does not. (O, G, V)

(5) If Oscar and George attend class, then either Miriam or Virginia attends. It is not the case that either George or Miriam attends class. Therefore, either Oscar or Virginia attends class. (O, G, M, V)

(6) If Oscar and George attend class, then so do Miriam and Virginia. It is not the case that George attends class only if Virginia attends class. Therefore, Oscar does not attend class. (O, G, M, V)

(7) A necessary condition for Oscar to attend class is that Miriam or Virginia attend. A sufficient condition for Virginia to attend class is that George attends. However, George does not attend unless Miriam attends, and Virginia attends only if Oscar attends. Therefore, Virginia attends class if and only if Oscar attends. (O, M, V, G)

(8) If Oscar attends class, then neither Miriam nor Virginia attend. If Virginia doesn't attend then neither does George, but if Miriam doesn't attend, then George does attend. Therefore, Oscar does not attend class. (O, M, V, G)

D. Construct a formal proof for each valid argument in part C. If the argument is not valid construct a counter-example. (Answers for 4 and 8.)

E. Determine the validity of each of the following arguments using truth tables. (Answers for 1, 4, and 8.)

(1) **Premises:** $O \rightarrow M \vee V, M, V \rightarrow O$

Conclusion: O

(2) **Premises:** $O \rightarrow (V \leftrightarrow G), \neg M \rightarrow (V \wedge \neg G)$

Conclusion: $\neg O \vee M$

(3) **Premises:** $O \rightarrow M, G \rightarrow V, \neg M \vee \neg V, G \vee \neg M$

Conclusion: $O \leftrightarrow \neg G$

(4) **Premises:** $O \rightarrow V, G \rightarrow M, G \rightarrow O \vee \neg M, G \vee \neg M, M \vee \neg V$

Conclusion: $O \leftrightarrow G$

(5) **Premises:** $O \wedge G \rightarrow V, V \rightarrow \neg M, \neg J \rightarrow M, M \rightarrow \neg J$

Conclusion: $G \rightarrow (O \rightarrow J)$

(6) **Premises:** $\neg O \rightarrow \neg V, O \rightarrow \neg G \vee M, \neg M$

Conclusion: $\neg G \vee \neg V$

(7) **Premises:** $(M \rightarrow O) \wedge (G \rightarrow V), M \vee G, O$

Conclusion: $O \wedge V$

(8) **Premises:** $M \vee V \rightarrow G, G \rightarrow V, O \rightarrow \neg J \vee \neg V$

Conclusion: $O \rightarrow (M \rightarrow \neg J)$

F. Construct a formal proof for each valid argument in part E. If the argument is not valid, construct a counter-example. (Answer for 4.)

G. Assign letters to the atomic sentences and translate each of the following arguments into logical notation. If the argument is valid, give a formal proof; otherwise give a counter-example. (Answers for 1, 5, 8(b), and 11.)

(1) If $a < b$ and $b < c$, then $a < c$. It is true that $a < b$, but it is not the case that $a < c$. Therefore, it is not the case that $b < c$.

(2) If $a > 0$, then a necessary and sufficient condition for $b > c$ is that $a \cdot b > a \cdot c$. It is true that $a \cdot b > a \cdot c$. Therefore, $b > c$.

In the remaining problems, we shall use the following abbreviations:

$x \not> y$ for it is not the case that $x > y$
$x \not< y$ for it is not the case that $x < y$
$x \neq y$ for it is not the case that $x = y$

(3) If $a > 0$ and $b > c$, then $a \cdot b > a \cdot c$, and if $a \not> 0$, but $b > c$, then $a \cdot b \not> a \cdot c$. Therefore, $a > 0$ if and only if $a \cdot b > a \cdot c$.

(4) If $a < 0$ and $0 < b$, then $a < b$. If $0 < c$ and $a < b$, then $a \cdot c < b \cdot c$. It is true that $a < 0$ and $0 < c$. Therefore, $0 < b$ only if $a \cdot c < b \cdot c$.

(5) A necessary and sufficient condition that $b < c$ is that $-c < -b$. Moreover, if $a + (-c) \not< a + (-b)$, then $b \not< c$. Therefore, $b < c$ if and only if $a + (-c) < a + (-b)$.

(6) If $a < 0$ and $b \not< 0$, then $a \cdot b \leq 0$. If $a \cdot b \leq 0$, then if $b < 0$, $a \not< 0$. If $a \cdot b \not\leq 0$, then either $a < 0$ and $b < 0$, or $a \not< 0$ and $b \not< 0$. If $a \cdot b \leq 0$ and $b \not< 0$, then $a < 0$. Therefore, a necessary and sufficient condition for $a \cdot b \leq 0$ is that $a < 0$ if and only if $b \not< 0$.

(7) Either $a < 0$ or $b < 0$. Also, $a \cdot b \leq 0$ if either $a < 0$ and $b \not< 0$, or if $b < 0$ and $a \not< 0$. Therefore, if not both $a < 0$ and $b < 0$, then $a \cdot b \leq 0$.

(8) (a). Define a function f so that

$$f(x) = \begin{cases} x, & \text{if } x < 0 \\ 5, & \text{if } x \not< 0. \end{cases}$$

Therefore, $f(x) = x$ or $f(x) = 5$. (*Hint:* Let $Z : x < 0$, $X : f(x) = x$, $F : f(x) = 5$.)

(b). Define a function g so that

$$g(x) = \begin{cases} f(x), & \text{if } x < 2 \\ 6, & \text{if } x \not< 2, \end{cases}$$

where $f(x)$ is defined in part (a). Assume $x < 0$ or $x \not< 2$. Therefore, $g(x) = x$ or $g(x) = 6$. (*Hint*: If $Z : x < 0$ and $T : x < 2$, then $0 \leq x < 2$ is $\neg Z \wedge T$. Let $X : g(x) = x$; $F : g(x) = 5$; $S : g(x) = 6$.)

(c). Let g be the function defined in part (b). Assume that neither does $g(x) = 5$ nor does $g(x) = 6$. Therefore, $g(x) = x$.

(9) If f is continuous on $(a, b]$, then f has a maximum on $[a, b]$ if f is defined at a. If f is not defined at a, then there is a point c between a and b such that $f(c) \geq f(x)$ for all $x \in [a, b]$. If there is such a point c between a and b such that $f(c) \geq f(x)$ for all $x \in [a, b]$, then f has a maximum on $[a, b]$.

(a) Therefore, f has a maximum on $[a, b]$.
(b) Therefore, if f is continuous on $(a, b]$, then f has a maximum on $[a, b]$.
(Use the letters C, M, D, and B for the atomic sentences.)

(10) A necessary condition for f to have a maximum on $[a, b]$ is that f is continuous on $(a, b]$ and f is defined at a. A necessary and sufficient condition for f to have a maximum on $[a, b]$ is that there is a point c between a and b such that $f(c) \geq f(x)$ for all $x \in [a, b]$. In fact, there is a point c between a and b such that $f(c) \geq f(x)$ for all $x \in [a, b]$. Therefore, f is continuous on $(a, b]$ and f is defined at a. (Use the same letters for atomic sentences as in problem 9.)

(11) If one and two are substituted for x in the inequality $x^2 + 1 > x + 1$, we obtain $2 > 2$ and $5 > 3$, respectively. However, $2 \not> 2$, but $5 > 3$. Therefore, one is not substituted for x in the inequality $x^2 + 1 > x + 1$, but two is. (Use A, B, $C: 2 > 2$, and $D: 5 > 3$ as the letters for the atomic sentences.)

(12) If one and two are substituted for x in the inequality $x^2 + 1 > x + 1$, we obtain $2 > 2$ and $5 > 3$, respectively. However, it is not the case that $2 > 2$, but $5 > 3$. Therefore, not both one and two are substituted for x in the inequality $x^2 + 1 > x + 1$. (Use the same letters for atomic sentences as in problem 11.)

H. Give a formal proof for each of the following valid arguments. (Answers for 1, 4, 8, and 10.)

(1) **Premises:** $A \to (B \to C)$, $C \to \neg D$, $\neg E \to D$
Conclusion: $B \to (A \to E)$

(2) **Premises:** $A \vee B$, $C \to \neg A$, $B \to D$, $C \to \neg D$
Conclusion: $\neg C$

(3) **Premises:** $(A \vee B) \vee C$, $A \to (D \leftrightarrow \neg E)$, $B \to \neg(\neg D \vee E)$, $E \to \neg(C \vee D)$, $\neg E \to C \wedge D$
Conclusion: $D \leftrightarrow \neg E$

(4) **Premises:** $(A \to B) \vee (A \wedge \neg C)$, A, $C \to \neg B$, $\neg C \to B$
Conclusion: $\neg C$

(5) **Premises:** $(A \to B) \wedge (C \to \neg D)$, $(E \to \neg B) \wedge (\neg F \to D)$, $\neg E \vee F \to G$, $\neg B \to D$, $A \vee C$
Conclusion: $B \wedge G$

(6) **Premises:** $A \wedge C \to D$, $B \wedge C \to D$, $\neg A \wedge \neg B \to E \vee F$, $G \to \neg E$, $F \to H$, $C \wedge \neg D$
Conclusion: $\neg G \vee H$

(7) **Premises:** $A \wedge (B \vee C)$, $A \wedge B \to D \wedge \neg F$, $A \to (C \to \neg(D \vee \neg F))$
Conclusion: $D \leftrightarrow \neg F$

(8) **Premises:** $A \wedge B \to C$, $A \vee E$, $G \to (\neg C \wedge \neg D)$, $A \leftrightarrow B$, $A \to C$, $C \to \neg D$, $B \to D$

Conclusion: $\neg A \wedge (G \to E)$

(9) **Premises:** $\neg A \to C \vee D$, $B \to E \wedge F$, $E \to D$, $\neg D$

Conclusion: $(A \to B) \to C$

(10) **Premises:** $A \to B$, $C \to \neg B$, $\neg C \to D$, $A \wedge \neg D$

Conclusion: $E \to F$

(11) **Premises:** $B \to A$, $\neg A \vee C$, $C \to D \vee E$, $D \to F \wedge \neg B$, $E \to \neg A \wedge F$, $F \to A$, $\neg B \vee G \to H \wedge I$, $H \to B$

Conclusion: $H \wedge \neg H$

(12) **Premises:** $A \to (B \to C)$, $\neg A \to D \wedge \neg E$, $A \wedge B \to \neg C$, $D \to F \vee G$, $\neg B \to (G \to H)$, $E \vee (H \vee \neg G)$

Conclusion: $G \to H$