MODULARITY OF COMPATIBLE FAMILY OF p-ADIC REPRESENTATIONS

1. Introduction

This notes proves the modularity of certain compatible family of l-adic Galois representations, via Serre and Kisin's arguments.

2. Compatible family of p-adic representations

Following [Tay06], we define that a *rank 2 weakly compatible system of* $\mathfrak{p}$*-adic representations* $\mathcal{R}$ over $\mathbb{Q}$ is a 5-tuple $(E,\, S,\, \{Q_l(X)\},\, \rho_{\mathfrak{p}},\, \{n_1, n_2\})$ where

- E is a number field over $\mathbb{Q}$;
- S is a finite set of primes over $\mathbb{Q}$;
- for each prime $l \notin S$, $Q_l(X)$ is a monic degree 2 polynomial in $E[X]$;
- for each prime $\mathfrak{p}$ of E, let p be the residue characteristic.

  $$\rho_{\mathfrak{p}} : G := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(E_{\mathfrak{p}})$$

  is a continuous representation such that, if $\mathfrak{p} \notin S$ then $\rho_{\mathfrak{p}}|_{G_p}$ is crystalline, and if $l \notin S$ and $l \neq p$ then $\rho_{\mathfrak{p}}$ is unramified at l and Fr_l has characteristic polynomial $Q_l(X)$;
- $\{n_1 n_2\}$ are integers such that for all primes $\mathfrak{p}$ of E (lying above a prime p) the representation $\rho_{\mathfrak{p}}|_{G_p}$ has Hodge-Tate weights n_1 and n_2.

Lemma 2.1. *Either all the $\rho_{\mathfrak{p}}$ is absolutely irreducible or all are absolutely reducible.*

Proof. Now suppose that $\rho_{\mathfrak{p}}$ is absolutely reducible and we want to show that for any other $\lambda \in \mathrm{Spec(E)}$, ρ_λ is also absolutely reducible. Note that there exists a finite extension K over $E_{\mathfrak{p}}$ such that ρ_p is reducible. Then there is a vector e_1 in the underline space $V' = V \otimes_{E_{\mathfrak{p}}} K$ such that G is stable over e_1. Let χ'_1 be the character of G acts on e_1, χ'_2 the character G acts on $V'/K \cdot e_1$. Since χ'_i is p-adic Hodge-Tate (i.e. potentially-semi-stable) character. Using Fontaine's classification, we can prove that $\chi'_i|_H \simeq \epsilon_p^{n'_i}$, where $H \subset I_p$ is an open subgroup. Since $\rho_{\mathfrak{p}}|_{G_p}$ has Hodge-Tate weights $n_1,\ n_2$. So we have no choice but $n_i = n'_i$ with $i = 1,\ 2$. Now $\chi_i = \chi'_i \epsilon_p^{-n_i}$ is a character of G such that χ_i has ramification at primes in $S \cup p$. Let L be the spitting field of χ_1. We claim that L must be a finite abelian extension of $\mathbb{Q}$. Let $L' \subset L$ be a finite abelian subfield. It suffices that we can bound the conductor of L'. There are two cases: Case I, let $l \in S$ and $l \neq p$. Set $I^w_{l,L'}$ the wild ramification group (i.e., the l-Sylow subgroup inside the ramification group $I_{l,L'}$). Then $i : I^w_{l,L'} \hookrightarrow \mathcal{O}^*_{E_{\mathfrak{p}}}$. We claim that $I^w_{l,L'} \hookrightarrow \mathcal{O}^*_{E_{\mathfrak{p}}}/1+\mathfrak{p}$. Suppose that $i(x) \in 1+\mathfrak{p}$. Note that $1+\mathfrak{p}$ is profinite p-group, but $i(x)$ has order l-power, thus $i(x) = 1$ and $I^w_{l,L'} \hookrightarrow \mathcal{O}^*_{E_{\mathfrak{p}}}/1+\mathfrak{p}$. Thus the order $I^w_{l,L'}$ is bounded. By [Ser] §4.9 Proposition 9, the conductor at l is bounded; Now considering the case II, conductor at p. The ramification index at p is $[I_p : H]$. Therefore we also bounded the conductor at p. In conclusion, we can bounded the conductor of L' and then the splitting field of

χ_i is finite. Therefore the images of χ_i are finite. Then there exists finite extension E'/E such that images of χ_i are inside $\mathcal{O}^*_{E'}$.

Now for any $\rho_{\mathfrak{q}}$ for $\mathfrak{q} \neq \mathfrak{p}$ and $\mathfrak{q}$ over rational prime q. Consider the $\mathfrak{q}$-adic representation $\rho'_{\mathfrak{q}} = \epsilon_q^{n_1}\chi_1 + \epsilon_q^{n_2}\chi_2$ defined over $E'_{\mathfrak{q}'}$, where prime $\mathfrak{q}' \in \mathrm{Spec}(E')$ over $\mathfrak{q}$. Since ϵ_p is compatible family of 1-dimensional p-adic Galois representation, the characteristic polynomial of Fr_l of $\rho'_{\mathfrak{q}}$ is the same as the that of $\rho_{\mathfrak{p}}$ for almost all primes l. Thus the characteristic polynomial of Fr_l of $\rho'_{\mathfrak{q}}$ is the same as that of $\rho_{\mathfrak{q}}$ for almost all primes l. Thus by Chebaratev density theorem, the traces of $\rho'_{\mathfrak{q}}$ and $\rho_{\mathfrak{q}}$ are the same. Then $\rho_{\mathfrak{q}}$ is reducible.

□

We call $\mathcal{R}$ *regular* if $n_1 < n_2$ and $\det\rho_{\mathfrak{p}}(c) = -1$ for one (and hence all) primes, where c denotes complex conjugation. Set $\epsilon = (\epsilon_p)$ the compatible system of p-adic cyclotomic characters. For any $i \in \mathbb{Z}$, denote $\mathcal{R}(i)$ the system $(\rho_{\mathfrak{p}}\epsilon_p^i)$.

Lemma 2.2. *$\mathcal{R}(-n_1)$ is weakly compatible system with Hodge-Tate weights 0 and $n_2 - n_1$.*

Proof. It suffices to show that for any $p \notin S$ and $l \notin S$ and $l \neq p$. The characteristic polynomial $f_l(X)$ of Fr_l is independent of choice of $\mathfrak{p}$. Note that $f_l(X) = \det(IX - \rho_{\mathfrak{p}}\epsilon_p^{-n_1}(\mathrm{Fr}_l)) = \det(IX - l^{-n_1}\rho_{\mathfrak{p}}((Fr)_l)) = l^{-2n_1}Q_l(l^{n_1}X)$. □

Now we state the classical theorem on compatible system constructed from modular forms. For any prime p, we fix an embedding $E \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $k \geq 2$, $N \geq 1$ and $S_k(\Gamma_1(N), \mathbb{C})$ the space of cuspidal modular form with weight k and level N. Suppose that $f = \sum_{i=1}^{\infty} a_n q^n$ is an eigenform normalized such that $a_1 = 1$.

Theorem 2.3. *Notations as above, then $E_f = \mathbb{Q}(a_n)_{n\geq 1} \subset \mathbb{C}$ is a number field. Moreover, for any $\lambda | p$ of E_f, there exists a continuous representation*

$$\rho_{f,\lambda} : G \longrightarrow \mathrm{GL}_2(E_{f,\lambda})$$

such that

1. *$\rho_{f,\lambda}$ is odd and absolutely irreducible.*
2. *For any $l \nmid Np$, $\rho_{f,\mathfrak{p}}$ is unramified over l and $\mathrm{tr}(\rho_{f,\lambda}(\mathrm{Fr}_l) = a_l$.*
3. *For any $\lambda | p$, $\rho_{f,\lambda}|_{G_p}$ is potential semi-stable with Hodge-Tate weights in $\{0, k-1\}$. If $\lambda \nmid N$, the $\rho_{f,\lambda}|_{G_p}$ is crystalline.*

Let $\rho_i : G \to \mathrm{GL}_2(E_i)$, $i = 1,\ 2$ be a two representations with E_i finite extensions of $\mathbb{Q}_p$ (resp. $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$). We write $\rho_1 \sim \rho_2$ if there exist an finite extension $E/\mathbb{Q}_p$ (resp. $E/\mathbb{F}_p$) such that $E_i \subset E$ for all $i = 1,\ 2$ and $\rho_1 \otimes_{E_1} E \simeq \rho_2 \otimes_{E_2} E$.

Theorem 2.4. *Let $\mathcal{R} = (\rho_{\mathfrak{p}})$ be an irreducible regular rank 2 weakly compatible system with weights $\{0, k\}$. Then there exists an eigenform f with weight $k+1$ such that for any $\rho_{\mathfrak{p}}$ there exists a prime λ of E_f satisfying $\rho_{\mathfrak{p}} \sim \rho_{f,\lambda}$.*

Note that all $\rho_{\mathfrak{p}}$ in Theorem 2.4 and $\rho_{f,\lambda}$ here are irreducible. To show they are isomorphic, using Chebotarev's density theorem, it suffices to show that there exists $\rho_{\mathfrak{p}}$, f and λ such that $\mathrm{tr}(\rho_{\mathfrak{p}}(\mathrm{Fr}_l)) = \mathrm{tr}(\rho_{f,\lambda}(\mathrm{Fr}_l))$ for almost all l.

Remark 2.5. The assumption on absolutely irreducibility here is equivalent to the following condition (pure weight k):

For any $l \notin S$ and for all $i : E \to \mathbb{C}$ the root of $i(Q_l(X))$ have absolute value $l^{k/2}$.

See Lemma 3.2 blew for the proof.

Corollary 2.6. *If $d = 1$ then the Scholl representation is modular.*

3. The proof of Theorem 2.4

In the proof, we mainly use Serre's conjecture. So let us first review Serre's conjecture. In the sequel, for any prime l, we use $G_l \subset G$ to denote the decomposition group over prime l and $I_l \subset G_l$ the inertia subgroup. $\mathbb{F}/\mathbb{F}_p$ is always a finite field with characteristic p.

3.1. **The strong Serre's conjecture.** In this subsection, we recall the precise form of Serre's conjecture which predicts not only that an odd representation $\bar{\rho} : G \to \mathrm{GL}_2(\mathbb{F})$ arises from a modular form, but also the minimal weight and level of the form.

Let

$$\omega_i : I_p \to \mathbb{F}_{p^i}^\times; \quad g \mapsto \frac{g(\sqrt[p^i-1]{p})}{\sqrt[p^i-1]{p}} \mod p$$

be the fundamental character of level i. We will write ω for ω_1, which is $\bmod p$ reduction of p-adic cyclotomic character ϵ_p.

Suppose we are given a representation $\bar{\rho}_p : G_p \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$. Then $\bar{\rho}_p|_{I_p}$ is either of the form $\begin{pmatrix} \omega^i & * \\ 0 & 1 \end{pmatrix} \otimes \omega^j$ with $i, j \in \mathbb{Z}$ or $\begin{pmatrix} \omega^i & 0 \\ 0 & \omega^{p^i} \end{pmatrix} \otimes \omega^j$ for some integers $i, j \in \mathbb{Z}$ and $p + 1 \nmid i$.

When $\bar{\rho}_p|_{I_p}$ is semi-simple, or equivalently tamely ramified, we can always choose $j \in [0, p-2]$ and $i+j \in [1, p-1]$; when $\bar{\rho}_p|_{I_p}$ is wildly ramified $i, j \in [0, p-2]$ can be uniquely determined. We set $k(\bar{\rho}_p) = 1 + i + (p+1)j$, unless $\bar{\rho}_p|_{I_p} \sim \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \otimes \omega^j$, with $*$ *très ramified*. In this exceptional case, we set $k(\bar{\rho}_p) = (p+1)(j+1)$.

For a representation $\bar{\rho} : G \to \mathrm{GL}_2(\mathbb{F})$, we set $k(\bar{\rho}) = k(\bar{\rho}|_{G_p})$ and set

$$N(\bar{\rho}) = \prod_{l \neq p} \mathrm{cond}(\bar{\rho}|_{G_l}),$$

where $\mathrm{cond}(\bar{\rho}|_{G_l})$ is the Artin conductor of $\bar{\rho}|_{G_l}$. Let V be the underlying space of $\bar{\rho}$, then $\mathrm{cond}(\bar{\rho}_{G_l}) = l^{n_l}$ where

$$n_l = \sum_{i=0}^{\infty} \frac{1}{(G_0 : G_i)} \dim(V/V^{G_i}) \tag{3.1}$$

where $G_i \subset G_0 = I_l$ are the ramification subgroups.

Theorem 3.1 (Serre's conjecture)**.** *Let $\bar{\rho} : G \to \mathrm{GL}_2(\mathbb{F})$ be odd and absolutely irreducible. Then there exists an eigenform f with weight $k(\bar{\rho})$ and level $N(\bar{\rho})$ such that $\bar{\rho} \sim \bar{\rho}_{f,\lambda}$.*

3.2. **The proof of Theorem 2.4.** Now we use Theorem 3.1 to prove the Theorem 2.4. We claim that there exist infinite many primes $\mathfrak{p}_i \in \mathrm{Spec}(E)$ such that

(1) $k(\bar{\rho}_{\mathfrak{p}_i}) = k+1$ for all i
(2) The set $\{N(\bar{\rho}_{\mathfrak{p}_i})\}$ is bounded.
(3) For all i, $\bar{\rho}_{\mathfrak{p}_i}$ is absolutely irreducible.

Let us first accept claim and prove Theorem 2.4. Suppose the set $\{\mathfrak{p}_i\}$ does exist. By Theorem 3.1, for any i, there exists an eigenform $f_i \in S_{k+1}(\Gamma_1(N(\bar{\rho}_{\mathfrak{p}_i})), \mathbb{C})$ such that $\bar{\rho}_{\mathfrak{p}_i} \sim \bar{\rho}_{f_i,\lambda_i}$. Select an N such that $N(\bar{\rho}_{\mathfrak{p}_i})|N$ for all i. We see that f_i are eigenforms in $S_{k+1}(\Gamma_1(N), \mathbb{C})$, which is a finite dimensional $\mathbb{C}$-space. So there are only finitely many normalized eigenforms. Therefore, there exists an eigenform f such that $f_i = f$ for infinitely many i. Without loss of generality, we assume that $f_i = f$ for all i.

Now for any fix prime $l \notin S$, let a_l be the coefficient of X in $Q_l(X)$. Since $\{\rho_{\mathfrak{p}}\}$ is compatible, for any $\mathfrak{p}_i \neq l$, $a_l = \mathrm{tr}(\rho_{\mathfrak{p}_i}(\mathrm{Fr}_l))$. On the other hand, set b_l the l-th Fourier coefficient of f. Then we have $b_l = \mathrm{tr}(\rho_{f,\lambda_i}(\mathrm{Fr}_l))$. Choose a Galois extension $F/\mathbb{Q}$ which contains E and E_f. Without loss of generality, we can assume our embedding $\iota : E \hookrightarrow F \hookrightarrow \bar{\mathbb{Q}}_p$ in a way such that $\mathfrak{p}_i = \mathcal{O}_E \cap \mathfrak{m}$ with $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{\bar{\mathbb{Q}}_p}$. Then λ_i is determined an embedding $\sigma_i : E_f \hookrightarrow F \hookrightarrow \bar{\mathbb{Q}}_p$. But there are only finitely many embeddings $E_f \hookrightarrow F$ here, so there must be an embedding σ such that $\sigma_i = \sigma$ for infinitely many i. Without loss of generality, we can assume that $\sigma = \sigma_i$ for all i, and we embed $E_f \to \bar{\mathbb{Q}}_p$ and $\lambda_i = E_f \cap \mathfrak{m}$. Set $q_i = F \cap \mathfrak{m}$.

Now since $\bar{\rho}_{\mathfrak{p}_i} \sim \bar{\rho}_{f,\lambda_i}$. Thus $a_l = b_l \mod q_i$ for all i. Since there are infinitely many i, we see that $a_l = b_l$ for all $l \notin S$. This prove Theorem 2.4.

3.3. **The proof of the claim.** The first two claims are not hard, while the last one need more work.

For any $p \notin S$ and $p > k+1$, we claim that for any $\mathfrak{p}|p$, $k(\bar{\rho}_{\mathfrak{p}}) = k+1$. In fact, since $\rho_{\mathfrak{p}}|_{G_p}$ is crystalline and Hodge-Tate weights are 0, k with $k \leq p-2$. The one can use Fontaine-Messing theory on strongly divisible lattices in filtered φ-modules to compute the reduction of such crystalline representations. Let T be a lattice in $\rho_{\mathfrak{p}}|_{G_p}$ and denote $\bar{T}$ the reduction of T. There are two cases:

Case I: T is irreducible, then $\bar{T}|_{I_p} \otimes \bar{\mathbb{F}}_p \sim \begin{pmatrix} \omega_2^k & 0 \\ 0 & \omega_2^{pk} \end{pmatrix}$. So $k(\bar{\rho}_{\mathfrak{p}}) = k+1$.

Case II: T is reducible, then $\bar{T}|_{I_p} \otimes \bar{\mathbb{F}}_p \sim \begin{pmatrix} \omega^k & * \\ 0 & 1 \end{pmatrix}$. In this case if $k > 1$, then we see that $k(\bar{\rho}_{\mathfrak{p}}) = k+1$. For $k = 1$, we must eliminate the case that $\bar{T}|_{I_p}$ is très ramifiée. And this case can be eliminated by some explicit computations.

Now let us bound the conductors of $\bar{\rho}_{\mathfrak{p}_i}$. We claim that there exist infinitely many primes $\mathfrak{p}_i \in \mathrm{Spec}(E)$ such that the set $\{N(\bar{\rho}_{\mathfrak{p}_i})\}$ is bounded. First note that for any $l \notin S$ and $l \neq p$, then $\rho_{\mathfrak{p}}$ is unramified at l. Therefore the conductor $N(\bar{\rho}_{\mathfrak{p}})$ only consists those primes in S. For any $l \in S$, let n_l be the integer defined in (3.1). Let F be the splitting field of $\bar{\rho}_{\mathfrak{p}}$ and G_0 the inertia subgroup at l inside $\mathrm{Gal}(F/\mathbb{Q})$ and G_1 the l-Sylow subgroup (i.e., teh wild inertia). Assume that the residue field $[k_{\mathfrak{p}} : \mathbb{F}_p] = g$. Note that $g \leq [E : \mathbb{Q}]$. Since that $G_0 \hookrightarrow \mathrm{GL}_2(k_{\mathfrak{p}})$. Thus we have $l^{m_l} := \#(G_1)|(p^{2g}-1)(p^{2g}-p^g)$. Thus $(p^{2g}-1)(p^g-1) = 0 \mod l^{m_l}$. Select an integer a_l sufficient large (depending on l and g) such that there exists a $b_l \in (\mathbb{Z}/l^{a_l}\mathbb{Z})^*$ satisfying $b_l^{2g} \neq 1 \mod l^{a_l}$. Therefore for any primes p such that

$p \equiv b_l \mod l^{a_l}$, $l^{a_l} \nmid (p^{2g}-1)(p^{2g}-p^g)$. Thus for any primes $\mathfrak{p}$ above p, $m_l < a_l$. Now by [Ser] §4.9 Proposition 9, so $n_l \leq 2(a_l + \frac{1}{l-1})$. Now by Chinese reminder theorem, we can select infinitely many primes p such that $p = b_l \mod l^{a_l}$. Thus there exists infinity many primes $\mathfrak{p}_i$ over those p such that $\{N(\bar{\rho}_{\mathfrak{p}_i})\}$ is bounded.

Lemma 3.2. *Let $\mathcal{R} = (\rho_{\mathfrak{p}})$ be a regular rank 2 weakly compatible system with weights $\{0, k\}$. Then $\mathcal{R}$ is absolutely irreducible if and only the following condition holds:*

For any $l \notin S$ and for all $i : E \to \mathbb{C}$ the root of $i(Q_l(X))$ have absolute value $l^{k/2}$.

Moreover, there exists for infinitely many prime $\mathfrak{p}_i$ such that $\bar{\rho}_{\mathfrak{p}_i}$ is absolutely irreducible.

Proof. If $\mathcal{R}$ is irreducible, by main theorem of [Tay06], we see that for any $l \notin S$ and for all $i : E \to \mathbb{C}$ the root of $i(Q_l(X))$ have absolute value $l^{k/2}$.

Now by the proof of the Lemma, we see that there exists a set S' of infinitely many primes $\mathfrak{p}_i$ such that $k(\bar{\rho}_{\mathfrak{p}_i}) = k+1$ and the set of $\{N(\bar{\rho}_{\mathfrak{p}_i})\}$ is bounded. Now we claim that there are only finitely many primes $\mathfrak{p}_i \in S'$ such that $\bar{\rho}_{\mathfrak{p}_i}$ is absolutely reducible. Now suppose that there exists a subset $S'' \in S'$ of infinitely many primes such that for any $\mathfrak{p}_i \in S''$ $\bar{\rho}_{\mathfrak{p}_i}$ is absolutely reducible. We would like to derive a contradiction.

Note that for any $p \geq k+2$, $p \notin S$ and $\mathfrak{p}|p$, then $T := \rho_{\mathfrak{p}}|_{G_p}$ is crystalline. As discussed in the beginning of this subsection, we have 2 cases of reduction, where the first case is absolutely irreducible. So if $\bar{\rho}_{\mathfrak{p}}$ is absolutely reducible, then we must have the second case. Therefore we have

$$\bar{\rho}_{\mathfrak{p}} \otimes \bar{\mathbb{F}}_p \sim \begin{pmatrix} \chi_1 \omega^k & * \\ 0 & \chi_2 \end{pmatrix} \tag{3.2}$$

where χ_1, χ_2 are characters unramified at p. It is easy to see that the conductor $N(\chi_j) | N(\bar{\rho}_{\mathfrak{p}})$ for $j = 1,\ 2$. We can lift χ_j to $\hat{\chi}_j : G \to \bar{\mathbb{Z}}^*$ with the same conductor. For any $\mathfrak{p}_i \in S''$, write $\hat{\chi}_j^{(i)}$ for characters attached to $\bar{\rho}_{\mathfrak{p}_i}$. Since the set of $\{N(\bar{\rho}_{\mathfrak{p}_i}),\ i \in S'\}$ is bounded, conductors $\hat{\chi}_j^{(i)}$ are bounded. So there are only finitely many $\hat{\chi}_j^{(i)}$. Therefore, without loss of generality, we can assume that $\hat{\chi}_j = \hat{\chi}_j^{(i)}$ for all i.

Now select a finite Galois extension F such that F contains E and all values of $\hat{\chi}_1$ and $\hat{\chi}_2$. Using the same trick in the second paragraph of the proof of Theorem 2.4, we can assume $\mathfrak{p}_i \in S''$ are all primes in $\mathcal{O}_F$. Now we claim that for any fixed prime $\mathfrak{q}|q$ of $\mathcal{O}_F$ and $\mathfrak{q} \in S''$, the semi-simplification $\rho_{\mathfrak{q}}$ is $\begin{pmatrix} \epsilon_q^k \hat{\chi}_1 & 0 \\ 0 & \hat{\chi}_2 \end{pmatrix}$, where ϵ_q is the q-adic cyclotomic character. It suffices to prove that their traces coincides for almost all primes. For any $l \notin S$ and $\mathfrak{q} \nmid l$, let $a_l = \mathrm{tr}(\rho_{\mathfrak{q}}(\mathrm{Fr}_l))$. For any $\mathfrak{p}_i \nmid l$, $\mathfrak{p}_i | p$ and $\mathfrak{p}_i \in S''$, we also have $a_l = \mathrm{tr}(\rho_{\mathfrak{p}_i}(\mathrm{Fr}_l))$. But $\bar{\rho}_{\mathfrak{p}_i}$ is reducible and has shape (3.2). So we have

$$a_l = \mathrm{tr}((\hat{\chi}_1 \epsilon_p^k + \hat{\chi}_2)(\mathrm{Fr}_l)) \mod \mathfrak{p}_i.$$

Note that (ϵ_p) is compatible system and there are infinitely many $\mathfrak{p}_i$. We have $a_l = \mathrm{tr}((\hat{\chi}_1 \epsilon_p^k + \hat{\chi}_2)(\mathrm{Fr}_l))$ and the semi-simplification $\rho_{\mathfrak{q}}$ is just $\begin{pmatrix} \epsilon_q^k \hat{\chi}_1 & 0 \\ 0 & \hat{\chi}_2 \end{pmatrix}$. This is impossible because the roots of $Q_l(X)$ have absolute value $l^{k/2}$ for all l. So except finitely many primes in S', $\bar{\rho}_{\mathfrak{p}}$ are absolutely irreducible. $\square$

4. Totally real case

Now let us extend some results of the above to the totally real case. Let F be a totally real field with $[F:\mathbb{Q}] = g$. Assume that F is Galois. A *regular* [1] *rank 2 weakly compatible system of λ-adic representations* $\mathcal{R}_F$ over F is a 5-tuple (E, S, $\{Q_{\mathfrak{l}}(X)\}$, ρ_λ, $\{n_1, n_2\}$) where

- E is a number field over F;
- S is a finite set of primes over F;
- for each prime $\mathfrak{l} \notin S$ of F, $Q_{\mathfrak{l}}(X)$ is a monic degree 2 polynomial in $E[X]$;
- for each prime λ of E, let p be the residue characteristic and $\mathfrak{p} := \lambda \cap \mathcal{O}_F$.
$$\rho_\lambda : G_F := \mathrm{Gal}(\overline{F}/F) \longrightarrow GL_2(E_\lambda)$$
is a continuous representation such that, if $\mathfrak{p} \notin S$ then $\rho_\lambda|_{G_{\mathfrak{p}}}$ is crystalline with the sets Hodge-Tate weights being $\{n_1, n_2\}$, and if $\mathfrak{l} \notin S$ then $\rho_\lambda|_{G_{\mathfrak{l}}}$ is unramified at $\mathfrak{l}$ and the Frobenius $\mathrm{Fr}_{\mathfrak{l}}$ has the characteristic polynomial $Q_{\mathfrak{l}}(X)$.
- $n_1 < n_2$ and $\det\rho_{\mathfrak{p}}(c) = -1$ for one (and hence all) primes, where c denotes complex conjugation.

Remark 4.1. In general, one should consider the case that the sets of Hodge-Tate weights are $\{n_1, \ldots, n_g\}$ in $\mathcal{R}$ instead of just $\{n_1, n_2\}$. But the definition of more general treatment is much more complicated. For our ad hoc concern of this note, we only consider the simpler case.

We call the compatible family $\mathcal{R}_F$ has a *semi-descent to* $\mathbb{Q}$ if for any prime $\lambda | p$ over $\mathcal{O}_E$ there exists a λ-adic representation $\rho'_\lambda : G_{\mathbb{Q}} \to GL_2(L_\lambda)$ and L_λ is a finite extension of E_λ such that $\rho'_\lambda|_{G_F} \sim \rho_\lambda$. Of course, if there exists a weakly compatible family $\mathcal{R}_{\mathbb{Q}} = (\rho'_\lambda)$ over $\mathbb{Q}$ such that $\rho'_\lambda|_{G_F} \sim \rho_\lambda$. Then we see that $\mathcal{R}_{\mathbb{Q}}$ is a quasi-descent of $\mathcal{R}_F$. In this case, we call that $\mathcal{R}_F$ has a *descent to* $\mathbb{Q}$.

Remark 4.2. In general, the descent of $\mathcal{R}_F$ is not unique.

Theorem 4.3. *Definitions as the above, let $\mathcal{R}_F = (\rho_\lambda)$ be a regular rank 2 weakly compatible family over F. Suppose that For any $\mathfrak{l} \notin S$ and for all $i : E \to \mathbb{C}$ the root of $i(Q_{\mathfrak{l}}(X))$ have absolute value $l^{\frac{n_2-n_1}{2}}$. Then $\mathcal{R}_F$ has a semi-descent to $\mathbb{Q}$ if and only if $\mathcal{R}_F$ has a descent $\mathcal{R}_{\mathbb{Q}}$ to $\mathbb{Q}$. In this case there exists a modular form f such that $\mathcal{R}_{\mathbb{Q}}$ comes from f.*

Of course if we know that $\mathcal{R}_F$ can be descent to a weakly compatible family $\mathcal{R}_{\mathbb{Q}}$ over $\mathbb{Q}$. Then the last statement is just the consequence of Theorem 2.4. But the proof of the Theorem actually circle around: we first prove that $\mathcal{R}_F$ comes from a modular form f over $\mathbb{Q}$. Then construct the descent $\mathcal{R}_{\mathbb{Q}}$ via f. Without loss of generality, we assume that $n_1 = 0$ and $n_2 = k$ from now on. To carry out the first step, we use the similar strategy of the proof of Theorem 2.4. So the following Lemma is crucial.

Lemma 4.4. *There exists a set $S' \subset \mathrm{Spec}(\mathcal{O}_E)$ of infinitely many primes such that*

(1) $k(\bar{\rho}'_\lambda) = k+1$ *for any* $\lambda \in S'$.

[1]need a better name here

(2) $\{N(\bar{\rho}'_\lambda)|\lambda \in S\}$ *is bounded.*
(3) $\bar{\rho}_\lambda$ *is absolutely irreducible for any* $\lambda \in S'$.

Proof. To prove the lemma, we first construct a set $\tilde{S}$ of infinitely many primes λ such that $\tilde{S}$ satisfies the first two requirements. For any prime $\lambda|p$ in $\mathcal{O}_E$, write $\mathfrak{p} = \lambda \cap F$. We claim that if $p \geq k+2$ and $\mathfrak{p} \notin S \cup \{p, p|\Delta_F\}$ where Δ_F is the discriminant of F, then $k(\bar{\rho}'_\lambda) = k+1$. To see this, for any such λ, consider $\rho'_\lambda|_{G_{\mathbb{Q}_p}}$. Note that $F_\mathfrak{p}$ is unramified extension over $\mathbb{Q}_p$, and $\rho_\lambda|_{G_{F_\mathfrak{p}}}$ is crystalline with Hodge-Tate weights $\{0, k\}$, so is $\rho'_\lambda|_{G_{\mathbb{Q}_p}}$. So the discussion of weights in the beginning of §3.3 applies, we get $k(\bar{\rho}'_\lambda) = k+1$.

Let us bound the conductor of $\bar{\rho}'_\lambda$. We only need show there exist infinitely many primes λ_i such that n_l defined in (3.1) are bounded for those rational primes l such that there is a prime $\mathfrak{l}|l$ and $\mathfrak{l} \in S$. Let L' be the splitting field of $\bar{\rho}'_\lambda$ and G'_0 the inertia subgroup at l inside $\mathrm{Gal}(L'/\mathbb{Q})$ and G'_1 the l-Sylow subgroup (i.e., the wild inertia subgroup). Write $l^{m'_l} = \#(G'_1)$. By [Ser] §4.9 Proposition 9, it suffices to bound m'_l for certain λ_i. Now consider $\bar{\rho}_\lambda$, Let L be the splitting field of $\bar{\rho}_\lambda$ and select $\mathfrak{l} \in \mathrm{Spec}(\mathcal{O}_F)$ is a prime over l. We also define G_0, G_1, m_i (respect to $\mathfrak{l}$) respectively. We claim that $m'_l \leq \log_l(g) + m_l$ and hence it suffices to bound m_l. To see the claim, note that $\bar{\rho}_\lambda \sim \bar{\rho}'_\lambda|_{G_F}$. Then $L' = LF$ and $[L' : L] \leq [F : \mathbb{Q}] = g$. Consequently $[G'_0 : G_0] \leq g$ and $[G'_1 : G_1] \leq g$. Hence $m'_l \leq m_l + \log_l(g)$. Now we can bound m_l just like $F = \mathbb{Q}$ case: Assume that the residue field $[k_\lambda : \mathbb{F}_p] = h$. Note that $h \leq [E : \mathbb{Q}]$. Since that $G_0 \hookrightarrow \mathrm{GL}_2(k_\lambda)$. Thus we have $l^{m_l} = \#(G_1)|(p^{2h}-1)(p^{2h}-p^h)$. Thus $(p^{2h}-1)(p^h-1) = 0 \mod l^{m_l}$. Select an integer a_l sufficient large (depending on l and h) such that there exists a $b_l \in (\mathbb{Z}/l^{a_l}\mathbb{Z})^*$ satisfying $b_l^{2h} \neq 1 \mod l^{a_l}$. Therefore for any primes p such that $p \equiv b_l \mod l^{a_l}$, $l^{a_l} \nmid (p^{2h}-1)(p^{2h}-p^h)$. Thus for any primes λ above p, $m_l < a_l$. Now by Chinese remainder theorem, there exists infinitely many primes λ_i over p_i such that $p_i \equiv b_i \mod l$. Hence $\{N(\bar{\rho}'_{\lambda_i})\}$ are bounded.

Now let us treat the last claim. By the proof above, we see that there exists a set S' of infinitely many primes λ_i such that $k(\bar{\rho}'_{\lambda_i}) = k+1$ and the set of $\{N(\bar{\rho}'_{\lambda_i})\}$ is bounded. Now we claim that there are only finitely many primes $\lambda_i \in S'$ such that $\bar{\rho}_{\lambda_i}$ (*Caution:* not only $\bar{\rho}'_\lambda$) is absolutely reducible. Now suppose that there exists a subset $S'' \in S'$ of infinitely many primes such that for any $\lambda_i \in S''$ $\bar{\rho}_{\lambda_i}$ is absolutely reducible. We would like to derive a contradiction.

Note that for any rational prime $p \geq k+2$, $\mathfrak{p} \notin S$ and $\lambda|p$ (recall that $\mathfrak{p} = \lambda \cap \mathcal{O}_F$), then $\rho_\lambda|_{G_{F_\mathfrak{p}}}$ is crystalline. If $p \nmid \Delta_F$ then we see that $\rho'_\lambda|_{G_{\mathbb{Q}_p}}$ is crystalline. Note we have two types of reductions for $\rho'_\lambda|_{G_{\mathbb{Q}_p}}$ as discussed in the beginning of §3.3. Either $\bar{\rho}'_\lambda|_{I_p} \otimes \bar{\mathbb{F}}_p \simeq \begin{pmatrix} \omega_2^k & 0 \\ 0 & \omega_2^{pk} \end{pmatrix}$ or $\bar{\rho}'_\lambda|_{I_p} \otimes \bar{\mathbb{F}}_p \sim \begin{pmatrix} \omega^k & * \\ 0 & 1 \end{pmatrix}$. If ρ'_λ has the first reduction type. Note that $F_\mathfrak{p}$ is unramified over $\mathbb{Q}_p$. Thus $\bar{\rho}'_\lambda|_{I_p} \sim \bar{\rho}_\lambda|_{I_\mathfrak{p}}$ and hence $\bar{\rho}_\lambda$ must be absolutely irreducible. So if λ is in S'', $\bar{\rho}'_\lambda|_{I_p}$ must have the second reduction type. Hence we have $\bar{\rho}'_\lambda \otimes \bar{\mathbb{F}}_p \sim \begin{pmatrix} \chi_1\omega^k & * \\ 0 & \chi_2 \end{pmatrix}$ where χ_1, χ_2 are characters unramified at all primes over p. Restricted to G_F, we have

$$\bar{\rho}_\lambda \otimes \bar{\mathbb{F}}_p \sim \begin{pmatrix} \chi_1\omega^k & * \\ 0 & \chi_2 \end{pmatrix} \tag{4.1}$$

where χ_1, χ_2 are characters unramified at all primes over p. It is easy to see that the conductor $N(\chi_j)|N(\bar{\rho}_\lambda)$ for $j = 1,\ 2$. We can lift χ_j to $\hat{\chi}_j : G_F \to \bar{\mathbb{Z}}^*$ with the same conductor. For any $\lambda_i \in S''$, write $\hat{\chi}_j^{(i)}$ for characters attached to $\bar{\rho}_{\lambda_i}$. Since the set of $\{N(\bar{\rho}_{\lambda_i}),\ i \in S'\}$ is bounded, conductors $\hat{\chi}_j^{(i)}$ are bounded. So there are only finitely many $\hat{\chi}_j^{(i)}$. Therefore, without loss of generality, we can assume that $\hat{\chi}_j = \hat{\chi}_j^{(i)}$ for all i.

Now select a finite Galois extension L such that L contains E and all values of $\hat{\chi}_1$ and $\hat{\chi}_2$. Using the same trick in the second paragraph of the proof of Theorem 2.4, we can assume $\lambda_i \in S''$ are all primes in $\mathcal{O}_L$. Now we claim that for any fixed prime $\mathfrak{q}|q$ of $\mathcal{O}_L$ and $\mathfrak{q} \in S''$, the semi-simplification $\rho_{\mathfrak{q}}$ is $\begin{pmatrix} \epsilon_q^k \hat{\chi}_1 & 0 \\ 0 & \hat{\chi}_2 \end{pmatrix}$, where ϵ_q is the q-adic cyclotomic character. It suffices to prove that their traces coincides for almost all primes. For any $\mathfrak{l}|l \notin S$ (l is a rational prime) and $\mathfrak{q} \nmid l$, let $a_{\mathfrak{l}} = \mathrm{tr}(\rho_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{l}}))$. For any $\lambda_i \nmid l$, $\lambda_i|p$ and $\lambda_i \in S''$, we also have $a_{\mathfrak{l}} = \mathrm{tr}(\rho_{\lambda_i}(\mathrm{Fr}_{\mathfrak{l}}))$. But $\bar{\rho}_{\lambda_i}$ is reducible and has shape (4.1). So we have

$$a_{\mathfrak{l}} = \mathrm{tr}((\hat{\chi}_1 \epsilon_p^k + \hat{\chi}_2)(\mathrm{Fr}_{\mathfrak{l}})) \mod \lambda_i.$$

Note that (ϵ_p) is compatible system and there are infinitely many λ_i. We have $a_{\mathfrak{l}} = \mathrm{tr}((\hat{\chi}_1 \epsilon_p^k + \hat{\chi}_2)(\mathrm{Fr}_{\mathfrak{l}}))$ and the semi-simplification $\rho_{\mathfrak{q}}$ is just $\begin{pmatrix} \epsilon_q^k \hat{\chi}_1 & 0 \\ 0 & \hat{\chi}_2 \end{pmatrix}$. This is impossible because the roots of $Q_{\mathfrak{l}}(X)$ have absolute value $l^{k/2}$ for all l. So except finitely many primes in S', $\bar{\rho}_{\lambda_i}$ are absolutely irreducible. $\square$

Proof of Theorem 4.3. By Lemma 4.4, we see that $\bar{\rho}'_{\lambda_i}$ is absolutely irreducible for all $\lambda_i \in S'$. Hence by Serre's conjecture (Theorem 3.1), there exists eigenform $f'_i \in S_{k+1}(\Gamma_1(N(\bar{\rho}'_{\lambda_i})), \mathbb{C})$ such that $\bar{\rho}_{f'_i, \alpha_i} \sim \bar{\rho}'_{\lambda_i}$, where α_i is a prime of E_{f_i} over p. Let f_i be the base change f'_i to F. We have $\bar{\rho}_{f_i, \alpha_i} \sim \bar{\rho}_{\lambda_i}$. Since f_i has the same weight and level as those f'_i. Now we just copy the remaining proof of Theorem 2.4. And we see that there exists an $f_i = f$ such that ρ_f gives the compatible family $\mathcal{R}_F$. Consequently, picking and f'_i such that $f_i = f$, then f'_i gives compatible family $\mathcal{R}_{\mathbb{Q}}$ over $\mathbb{Q}$ which is a descent of $\mathcal{R}_F$. $\square$

5. Descend a p-adic Galois representation

In this section, let us discuss an intrinsic condition to descend a p-adic Galois representation. Let F, K be number fields and F/K Galois, $g = [F : \mathbb{Q}]$, $E/\mathbb{Q}_p$ a finite extension and $\rho : G_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Aut}_E(V)$ a p-adic Galois representation. We say ρ satisfies *quasi-descent* condition if the following holds:

There exists a finite set of primes $S_\rho \subset \mathrm{Spec}(\mathcal{O}_K)$ such that

(1) S_ρ contains all ramified primes of F/K and ρ.
(2) For any primes $\mathfrak{l}$ of $\mathcal{O}_K$ such that $\mathfrak{l} \notin S_\rho \cup \{p\}$, write $\mathfrak{l} = (\wp_1 \wp_2 \dots \wp_m)^e$ in $\mathcal{O}_F$, then

$$\det(\lambda I - \rho(\mathrm{Fr}_{\wp_i})) = \det(\lambda I - \rho(\mathrm{Fr}_{\wp_j})) \text{ for any } i, j = 1, \dots, m.$$

Apparently, suppose that there exists a p-adic representation $\rho' : G_K \to \mathrm{Aut}_{E'}(V')$ such that $\rho'|_{G_F} \sim \rho$ then ρ satisfies the quasi-descent condition. In this case, we call ρ' is a *descent* of ρ. Conversely, we have the following question:

Question 5.1. *Is that true that ρ satisfies the quasi-descent condition if ρ has a descent?*

Remark 5.1. In general, V and V' are not defined in the same coefficients fields, even if the descent exists. Here is an example: let $p = 3$, select a cyclic extension $F/\mathbb{Q}$ with degree g such that there exists a g-th roots of unity not in $\mathbb{Z}_3$, and $E/\mathbb{Z}_3$ a finite extension such that g-th roots of unity are all in $\mathcal{O}_E$. Consider the Galois character $\rho' : G_{\mathbb{Q}} \to \mathrm{Gal}(F/\mathbb{Q}) \to \mathcal{O}_E^*$ by sending the generator of $\mathrm{Gal}(F/\mathbb{Q})$ to the g-th root of unity. $\rho = \rho'|_{G_F}$ is a trivial representation. So ρ can be defined over $\mathbb{Q}_3$. This example also shows that given ρ the descent may not be unique. In fact, any character of $\mathrm{Gal}(F/\mathbb{Q})$ can be a descent of ρ.

However we can classify all descents of ρ in some nice situations. Let $\sigma \in G_K$, in the sequel, we write $\rho^\sigma : G_F \to \mathrm{Aut}_E(V)$ such that for any $\tau \in G_F$ and $v \in V$, $\rho^\sigma(\tau)v = \rho(\sigma^{-1}\tau\sigma)v$. Sometime, we may just use V^σ to denote ρ^σ. Note that if ρ has a descent ρ'. Then $\rho'(\sigma) : V \to V$ induces an isomorphism $\rho \to \rho^\sigma$.

Proposition 5.2. *Suppose that F/K is cyclic and ρ is absolutely irreducible. Assume that ρ' is a descent of ρ. Let $\hat{H}$ be the group of all E-characters of* $\mathrm{Gal}(F/K)$. *Then $\{\rho' \otimes \chi | \chi \in \hat{H}\}$ exhausts all possible descents of ρ.*

Proof. Let $\sigma \in G_K$ such that σ is a generator of $\mathrm{Gal}(F/K)$. Assume that $\rho'' : G_K \to \mathrm{Aut}_{E'}(V')$ be another descent of ρ. We need show that there exists a $\chi \in \hat{H}$ such that $\rho'' \sim \rho' \otimes \chi$. Without loss of generality, we can assume that all representations here are finite dimensional $\overline{\mathbb{Q}}_p$-spaces. Note that both $\alpha' := \rho'(\sigma)$ and $\alpha'' := \rho''(\sigma)$ induce isomorphisms $\rho \to \rho^\sigma$. Hence $(\alpha'')(\alpha')^{-1} \in \mathrm{Aut}_{\overline{\mathbb{Q}}_p[G_F]}(\rho)$. Since ρ is absolutely irreducible, we have $\mathrm{Aut}_{\overline{\mathbb{Q}}_p[G_F]}(\rho) = \overline{\mathbb{Q}}_p$. Thus there exists a constant $\zeta \in \overline{\mathbb{Q}}_p$ such that $\alpha'' = \zeta\alpha'$. On the other hand, since $\sigma^g \in G_F$, we have $(\rho''(\sigma))^g = (\rho'(\sigma))^g$. Hence $\zeta^g = 1$ and ζ is a g-th root of unity. Let $\chi \in \hat{H}$ such that $\chi(\sigma) = \zeta$ and we see that $\rho'' \sim \rho' \otimes \chi$. □

Now let us gives a partial answer to Question 5.1.

Proposition 5.3. *Assume that F/K is cyclic and ρ is absolutely irreducible. Then Conjecture 5.1 is true.*

Proof. Without loss of generality, we assume that $E = \overline{\mathbb{Q}}_p$. Select a $\sigma \in G_K$ such that σ is a generator of $\mathrm{Gal}(F/K)$. The quasi-descent conditions implies that there exists an isomorphism $f_\sigma : V^\sigma \to V$. Then it is easy to check that $(f_\sigma)^g : V^{\sigma^g} \to V$ is an isomorphism. On the other hand, $f' := \rho(\sigma^g)$ induces to isomorphism $V^{\sigma^g} \to V$. So $(f')(f_\sigma)^{-g} : V \to V$ is inside $\mathrm{Aut}_{\overline{\mathbb{Q}}_p[G_F]}(V)$, which is $\overline{\mathbb{Q}}_p$ by the absolutely irreducibility of ρ. So there exists an $a \in \overline{\mathbb{Q}}_p$ such that $(f_\sigma)^g a^g = f'$. So after replace f_σ by $f_\sigma a$, we can assume that $f_\sigma^g = f' = \rho(\sigma^g)$.

Now let us construct ρ' as following: For any $\tau \in G_K$, τ can be written uniquely $\tau = \sigma^m\beta$ with $0 \le m < g$ and $\beta \in G_F$. Set $\rho'(\tau) = (f_\sigma)^m\rho(\beta)$. Now it suffices to check that $\rho'(\tau_1\tau_2) = \rho'(\tau_1)\rho'(\tau_2)$. Now write $\tau_i = \sigma^{m_i}\beta_i$. We have

$$\rho'(\tau_1)\rho'(\tau_2) = ((f_\sigma)^{m_1}\rho(\beta_1))((f_\sigma)^{m_2}\rho(\beta_2)) = (f_\sigma)^{m_1+m_2}((f_\sigma)^{-m_2}\rho(\beta_1)(f_\sigma)^{m_2})\rho(\beta_2).$$

One the other hand,

$$\tau_1\tau_2 = \sigma^{m_1+m_2}\sigma^{-m_2}\beta_1\sigma^{m_2}\beta_2 = \sigma^{m'}\sigma^{qg}(\sigma^{-m_2}\beta_1\sigma^{m_2})\beta_2$$

where $m_1 + m_2 = m' + qg$ with $0 \leq m' < g$. Hence

$$\rho'(\tau_1\tau_2) = (f_\sigma)^{m'}\rho(\sigma^{qg})\rho(\sigma^{-m_2}\beta_1\sigma^{m_2})\rho(\beta_2)$$

Now we need to check $(f_\sigma)^{m_1+m_2} = (f_\sigma)^{m'}\rho(\sigma^{qg})$ and $(f_\sigma)^{-m_2}\rho(\beta_1)(f_\sigma)^{m_2} = \rho(\sigma^{-m_2}\beta_1\sigma^{m_2})$. But these follows the facts that $f_\sigma^g = \rho(\sigma^g)$ and f_σ is an isomorphism $V^\sigma \to V$. □

Now let $\mathcal{R}_F = (\rho_\lambda)$ is a regular weakly compatible system over a Galois totally real field F as defined in §4. Suppose that one representation ρ_β satisfies the quasi-descent condition. Since (ρ_λ) is a compatible family, all ρ_λ satisfies the quasi-descent. Now combining Proposition 5.3 and theorem 4.3 together, we have

Theorem 5.4. *Let $\mathcal{R}_F = (\rho_\lambda)$ be a regular weakly compatible system of λ-adic Galois representations over a totally real field F. Suppose the following holds:*

1. *$F/\mathbb{Q}$ is cyclic.*
2. *One of ρ_λ satisfies the quasi-descent condition.*
3. *For any $\mathfrak{l} \notin S$ and for all $i : E \to \mathbb{C}$ the root of $i(Q_\mathfrak{l}(X))$ have absolute value $l^{\frac{n_2-n_1}{2}}$.*
4. *Except finitely many primes, ρ_λ are absolutely irreducible.*

Remark 5.5. It seems that one can relax (1) to case that F is solvable and (4) seems to be removable. Furthermore, (3) also can be removed if one can extend the main results of [Tay06]to totally real field. But all these seems need some non-trivial efforts.

Corollary 5.6. *Let ρ be a p-adic Galois representation of $G_\mathbb{Q}$. Suppose that there exists a totally real field $F/\mathbb{Q}$ such that ρ restricted to G_F comes from a Hilbert modular form f over F and $F/\mathbb{Q}$ is cyclic. Then ρ comes from a modular form.*

Proof. f gives arise a regular compatible family $\mathcal{R}_F$ which satisfies (3) and (4) in the above theorem. Since ρ is a descent of $\rho|_{G_F}$, ρ satisfies the quasi-descent condition. Then the corollary follows. □

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