INTRODUCTION TO p-ADIC HODGE THEORY

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1. Introduction

Fix l a prime. Let $G := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ the absolute Galois group. A l-adic Galois representations (of G) is a continuous linear representation of G on a finite dimensional $\mathbb{Q}_l$-vector space. The following examples of l-adic Galois representations are important:

Example 1.1 (l-adic cyclotomic character)**.** Let μ_{l^n} be the group of l^n-th roots of unity. G acts on μ_{l^n}, hence on $\varprojlim_n \mu_{l^n} \simeq \mathbb{Z}_l$. Therefore, we get 1 dimensional l-adic Galois representation: $\epsilon : G \to \mathrm{Aut}_{\mathbb{Z}_l}(\mathbb{Z}_l) = \mathbb{Z}_l^*$, the l-adic cyclotomic character.

Example 1.2 (l-adic Tate module)**.** Let A be an abelian variety over $\mathbb{Q}$ of dimension g. Let $A[l^n] := \{x \in A(\bar{\mathbb{Q}}_l) | l^n x = 0\}$. Then $A[l^n] \simeq \bigoplus_{i=1}^{2g} \mathbb{Z}/l^n\mathbb{Z}$ and G acts on $A[l^n]$. Therefore G acts on the l-adic Tate module $T_l(A) = \varprojlim_n A[l^n]$. Then $V_l(A) := T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is $2g$ dimensional l-adic Galois representation.

Let p be a prime and G_p a decomposition group at p. We may identify G_p with the local Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ at p. We always have the exact sequence:

$$1 \to I_p \to G_p \to \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \to 0.$$

Let $\rho : G \to \mathrm{Aut}_{\mathbb{Q}_l}(V)$ be a l-adic Galois representation. We call ρ is *unramified at p* if $\rho(I_p)$ is trivial.

Example 1.3 (modular representations)**.** Let $f \in S_k(\Gamma(N))$ be a newform with weight k and level N. Suppose that the Fourier expansion is $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z/N}$ with $a_n = 1$. Then for any l, one[1] can construct a l-adic representation $\rho_{f,l} : G \to \mathrm{GL}_2(E)$, where E is a finite extension of $\mathbb{Q}_l$ containing all a_n. For any $p \nmid Nl$, $\rho_{f,l}$ is unramified at p and $\mathrm{trace}(\rho_{f,l})(\mathrm{Fr}_p) = a_p$ where Fr_p is the Frobenius at p.

Example 1.4 (l-adic cohomology)**.** Let X be an algebraic variety over $\mathbb{Q}$. We can define l-adic étale cohomology [2] $\mathrm{H}^i_{\text{ét}}(X \otimes \bar{\mathbb{Q}}, \mathbb{Z}_l)$ and $\mathrm{H}^i_{\text{ét}}(X \otimes \bar{\mathbb{Q}}, \mathbb{Q}_l) := \mathrm{H}^i_{\text{ét}}(X \otimes \bar{\mathbb{Q}}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Example 1.1, 1.2, are the dual of the l-adic étale cohomology where $i = 1$ and X is respectively the multiplicative group $\mathbb{G}_m$ and abelian varieties. Example 1.3, are essentially comes from an l-adic étale cohomology via a much more dedicated construction.

[1] The construction is highly non-trivial, see [Con] for details. In general, it is still an open question to construct a l-adic Galois representation from a general automorphic form

[2] See SGA 4 or Milne's book: *Lectures on Etale Cohomology*, available at `http://www.jmilne.org/math/index.html`

All our examples of l-adic representations come from geometry. One of the key questions in this area is that how we can describe those l-adic Galois representations from algebraic geometry by purely Galois representation theoretic language.

If ρ is a l-adic representation from geometry, then one can prove that ρ is unramified except finitely many places. This is mainly due to the fact that a variety over $\mathbb{Q}$ always has an integral model over $\mathbb{Z}$ which is smooth except finitely many places. In fact, for any $p \neq l$, one can prove that $\rho|_{G_p}$ is not far from being unramified. (See the fourth lecture in details).

On the other hand, when $l = p$, $\rho_p := \rho|_{G_p}$ can be much more complicated even if ρ comes from geometry. In particular, ρ_p can be very ramified, i.e., $\rho(I_p)$ has very huge image (For an example, p-adic cyclotomic character). To deal with this situation, Fontaine invent the p-adic theory to classified those p-adic representations from Geometry.

In general, *p-adic representations* always refer to those continuous representations of G_p to finite dimensional $\mathbb{Q}_p$-vector spaces.

A p-adic representation V of G_p is called *semi-stable* if $\dim_{\mathbb{Q}_p}(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_p} = \dim_{\mathbb{Q}_p} V$, where B_{st} is a huge $\mathbb{Q}_p$-algebra with a continuous G_p-action and other extra structures. The construction of B_{st} (See also [Fon94]) and the classification of semi-stable representation (by admissible filtered (φ, N)-module) consists the main body of this mini course[3].

A p-adic Galois representation ρ is called *potentially semi-stable* if there exists a finite extension $K/\mathbb{Q}_p$ such that ρ restricted to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ is semi-stable. By Tsuji's comparison theorem ([Tsu99]), we know that p-adic étale cohomology $\mathrm{H}^i(X \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_p)$ is always potentially semi-stable. Hence any p-adic representation from geometry is potentially semi-stable. In particular, if ρ is the l-adic representations from Example 1.1, 1.2 and 1.3, $\rho|_{G_p}$ is potentially semi-stable. Therefore Fontaine and Mazur ([FM95]) made the following remarkable conjecture:

Conjecture 1.5 (Fontaine-Mazur). *(1) Let $\rho : G \to \mathrm{GL}_n(\mathbb{Q}_l)$ be a continuous representation, which is irreducible, unramified away from finitely many primes, and whose restriction to every decomposition group over l is potentially semi-stable. Then ρ is geometric in the sense that, up to a twist, it appears as a subquotient in the étale cohomology of a finite type $\mathbb{Q}$-scheme.*

(2) Let $E/\mathbb{Q}_l$ be a finite extension. Suppose that $\rho : G \to \mathrm{GL}_2(E)$ is a continuous representation, which is odd, irreducible, unramified outside finitely many prime, and whose restriction to G_l is potentially semi-stable, when regard as a $\mathbb{Q}_l$-representation. Then up to a twist of ϵ, it arises from a modular form, i.e., there exists a modular form f such that $\rho \simeq \rho_{f,l}$.

Leading by the breaking through work of Wiles, many cases of Conjecture 1.5 have been known and we have many important applications, e.g., the proof of Fermat's last theorem, Sato-Tate conjecture and Serre's modularity conjecture. [4] In the proof of many known cases (especially in work of Kisin), p-adic Hodge theory

[3]Of course, we will also discuss crystalline representations and Hodge-Tate representations

[4]Note that Taniyama-Shimura conjecture is the only the special case of Conjecture 1.5 for $k = 2$ where k is the weight of modular forms.

plays a cental technical role. Here we cite one of the most recent results from Kisin[5] which depends on the known case of p-adic Langlands correspondence:

Theorem 1.6 ([Kis06]). *Fix a prime $p > 2$, S a set of prime containing p, ∞, $G_{\mathbb{Q},S}$ the Galois group of maximal unramified extension of $\mathbb{Q}$ outside of S and G_p the decomposition group at p. Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$, having the residue field $\mathbb{F}$, and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(O)$ a continuous representation. Suppose that*

(1) $\rho|_{G_p}$ *is potentially semi-stable with distinct Hodge-Tate weights.*
(2) $\rho|_{G_p}$ *is semi-stable over an abelian extension over $\mathbb{Q}_p$.*
(3) $\bar{\rho} :\xrightarrow{\rho} \mathrm{GL}_2(O) \to \mathrm{GL}_2(\mathbb{F})$ *is modular, and $\bar{\rho}|_{\mathbb{Q}(\zeta_p)}$ is absolutely irreducible.*
(4) $\bar{\rho}|_{G_p} \not\sim \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$ *for any character $\chi : G_p \to \mathbb{F}^\times$, where $\omega := \epsilon \mod p$.*

Then up to a twist, ρ is modular.

Variations of the above Theorem on potentially Barsoti-Tate representations are some essential inputs in Khare and Wintenberger to prove Serre's modularity conjecture:

Theorem 1.7. *Notations as in Theorem 1.6, Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation with odd determinant then $\bar{\rho}$ is modular.*

Injecting Theorem 1.7 into Theorem 1.6, the condition that $\bar{\rho}$ is modular then can be removed.

2. References

2.1. General discussion on Galois representations.

(1) Jean-Pierre Serre, *Abelian l-adic representations and Elliptic curves.*
(2) Jean-Marc Fontaine, Yi Ouyang, *p-adic Galois representations*, available at `http://faculty.math.tsinghua.edu.cn/faculty/~ youyang/`

2.2. Survey on p-adic Hodge theory.

(1) Laurent Berger, *An Introduction to theory of p-adic representations*, available in *arxiv.*

The references for the following topics are numerous, but we only list those that are important for beginners.

2.3. Classical Theory on Big rings and weakly admissible (φ, N)-modules.

(1) Astérisque 223, *Périods p-adiques*, Exposé II, III, VIII.
(2) [CF00]

2.4. Comparison Theorems, which can explain why p-adic representations from geometry is potentially semi-stable.

(1) [FM87]
(2) [Tsu99]

[5] Note that this celebrating result does not cover all known cases of Conjecture 1.5.

2.5. Theory of (φ, Γ)-module.

(1) [Fon90]
(2) Pierre Colmez, *Fontaine's ring and p-adic L-functions*, available at `http://faculty.math.tsinghua.edu.cn/faculty/~ youyang/`

2.6. Integral p-adic Hodge theory.

(1) [Bre02]
(2) [Liu]

2.7. Theory of Modularity and p-adic Hodge theory. There are too many (and difficult) papers in this direction. One can find some relative friendly papers in Kisin and Breuil's homepages. For example:

(1) Mark Kisin, *Modularity of some geometric Galois representations.*
(2) Mark Kisin, *Modularity of 2-dimensional Galois representation*
(3) Christopher Breuil, *Towards a p-adic Langlands programme*[6]

Of course, the best book to explain the original idea of Wiles is *Modular Forms and Fermat's Last Theorem* by Silverman, Cornell, and Stevens.

References

[Bre02] Christophe Breuil, *Integral p-adic Hodge theory*, Algebraic geometry 2000, Azumino (Hotaka), Adv. Stud. Pure Math., vol. 36, Math. Soc. Japan, Tokyo, 2002, pp. 51–80.
[CF00] Pierre Colmez and Jean-Marc Fontaine, *Construction des représentations p-adiques semi-stables*, Invent. Math. **140** (2000), no. 1, 1–43.
[Con] Brian Conrad, *Modular forms and the ramanujan conjecture*, available soon, Available at `bdconrad@umich.edu`.
[FM87] Jean-Marc Fontaine and William Messing, *p-adic periods and p-adic étale cohomology*, Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), Contemp. Math., vol. 67, Amer. Math. Soc., Providence, RI, 1987, pp. 179–207.
[FM95] Jean-Marc Fontaine and Barry Mazur, *Geometric Galois representations*, Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993), Ser. Number Theory, I, Internat. Press, Cambridge, MA, 1995, pp. 41–78.
[Fon90] Jean-Marc Fontaine, *Représentations p-adiques des corps locaux. I*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 249–309.
[Fon94] ———, *Le corps des périodes p-adiques*, Astérisque (1994), no. 223, 59–111.
[Kis06] Mark Kisin, *The fontaine-mazur conjecture for* GL_2, Preprint (2006).
[Liu] Tong Liu, *Main conjecture of integral p-adic Hodge theory*, Preprint, avaliable at `http://www.math.upenn.edu/~ tongliu/research.html`.
[Tsu99] Takeshi Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. **137** (1999), no. 2, 233–411.

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[6]This paper discuss basic idea of p-adic Langlands which essentially used in the proof of Theorem 1.6