POTENTIALLY GOOD REDUCTION OF BARSOTTI-TATE GROUPS

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ABSTRACT. Let R be a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field, K the fraction field of R. Suppose G is a Barsotti-Tate group (p-divisible group) defined over K which acquires good reduction over a finite extension K' of K. We prove that there exists a constant $c \geq 2$ which depends on the absolute ramification index $e(K'/\mathbb{Q}_p)$ and the height of G such that G has good reduction over K if and only if $G[p^c]$ can be extended to a finite flat group scheme over R. For abelian varieties with potentially good reduction, this result generalizes Grothendieck's "p-adic Néron-Ogg-Shafarevich criterion" to finite level. We use methods that can be generalized to study semi-stable p-adic Galois representations with general Hodge-Tate weights, and in particular leads to a proof of a conjecture of Fontaine and gives a constant c as above that is independent of the height of G.

CONTENTS

1. INTRODUCTION

Let R be a complete discrete valuation ring of mixed characteristic $(0, p)$, K the fraction field, $\mathfrak{m}$ the maximal ideal of R, $k = R/\mathfrak{m}$ the residue field and A an abelian variety defined over K. We say A has *good reduction* if there exists an abelian scheme $\mathscr{A}$ over R such that $\mathscr{A} \otimes_R K \simeq A$. We say A has *potentially good reduction* if there exists a finite extension K'/K and an abelian scheme $\mathscr{A}'$ over R' such that $\mathscr{A}' \otimes_{R'} K' \simeq A \otimes_K K'$. Let I be the inertia subgroup of $\mathrm{Gal}(\overline{K}/K)$ and $l \neq p$ be any prime. Using the criterion of Néron-Ogg-Shafarevich [16], it is not difficult to prove that if A has potentially good reduction then A has good reduction if and only if the action of I on $A[l^c]$ is trivial, where $c = 1$ for $l \neq 2$ and $c = 2$ for $l = 2$.

When $l = p$, the problem is more subtle. For example, the relevant finite flat group schemes are not étale, and thus may not be determined by their generic fiber as in the case $l \neq p$. Work of Grothendieck [1] and Raynaud [15] proves that A has good reduction if and only if for all n, $A[p^n]$ can be extended to a finite flat group scheme over R. In particular, the property of good reduction (and hence potentially good reduction) is encoded in the Barsotti-Tate group $G = \varinjlim A[p^n]$ (also known as the p-divisible group associated to A). This fact is implicit in the work of Wiles and others on modularity of certain Galois representations of weight 2 ([5], [19]). In what follows, we fix $l = p$ and study potentially good reduction for Barsotti-Tate groups.

Let K' be a finite extension of K, R' the ring of integers of K'. Suppose a Barsotti-Tate group G with height h defined over K has *good reduction* over K' in the sense that there exists a Barsotti-Tate group $\mathcal{G}'$ over R' such that $\mathcal{G}' \otimes_{R'} K' \simeq G \otimes_K K'$. By §2, [15], G has good reduction over K if only if $G[p^n]$ can be extended to a finite flat group scheme over R for all n. The following question was raised by N. Katz:

For G as above with height h and good reduction over K', does there exist a constant $c = c(K', K, h)$ such that G has good reduction over K if and only if $G[p^c]$ can be extended to a finite flat group scheme over R? If so, how does c depend on the arithmetic of K'/K, and how does it depend on the height h?

The general case is easily reduced to the case when K'/K is totally ramified. Let e be the absolute ramification index of $K'/\mathbb{Q}_p$. In the case $e < p-1$, B. Conrad solved this question in [7]. He proved that if $G[p]$ can be extended to a finite flat group scheme, then G has good reduction. That is, when $e < p-1$, c can be taken to be 1 for all h. His idea was to compute descent data for the generalized Honda system associated to G. Unfortunately, in the case $e \geq p-1$, there is no Honda system available. Although the theory of Breuil modules [4] is available in this case (at least for $p > 2$), its behavior under base change is very complicated due to the dependence of theory on the choice of uniformizer, so it is hard to apply Breuil's theory for this question (especially when $p \mid e$).

We use Messing's deformation theory [14] to attack this question and get the following result:

Theorem 1.0.1 (Main Theorem). *Assume that k is perfect. Suppose G is a Barsotti-Tate group over K that acquires good reduction over a finite extension K'. There exists a constant $c \geq 2$ which depends on $e = e(K'/\mathbb{Q}_p)$ and the height*

of G such that G has good reduction over R if and only if $G[p^c]$ can be extended to a finite flat group scheme over R.

As we will explain in Remark 1.0.7, a refinement of the methods of this paper allows us to take c to be independent of the height of G (though in practice one usually studies G's with a fixed, or at least bounded, height). Applying this result to the case of abelian varieties and using the precise form of the semi-stable reduction theorem for abelian varieties (i.e., bounding the degree of the extension over which semi-stable reduction is attained), we get :

Theorem 1.0.2. *Assume that k is perfect. Suppose A is an abelian variety over K with potentially good reduction. There exists a constant $c' \geq 2$ depending on $e(K/\mathbb{Q}_p)$ and $\dim A$ such that A has good reduction over K if and only if $A[p^{c'}]$ can be extended to a finite flat group scheme over R.*

We may and do assume R'/R is totally ramified and generically Galois. Let $\Gamma = \mathrm{Gal}(K'/K)$ and $G' = G \otimes_K K'$. Since G' has good reduction over K', there exists a Barsotti-Tate group $\mathcal{G}'$ over R' such that $\mathcal{G}' \otimes_{R'} K' \simeq G'$. Using the K-descent G of G', clearly G' has natural descent data ϕ_σ, where $\phi_\sigma : \sigma^*(G') \simeq G'$ is a collection of K'-automorphisms satisfying the cocycle condition. By Tate's Theorem [18], the descent data on the generic fiber will uniquely extend over R'. That is, we get isomorphisms $\widetilde{\phi}_\sigma : \sigma^*(\mathcal{G}') \simeq \mathcal{G}'$ over R' satisfying the cocycle condition, but this is *not* fppf descent data with respect to $R \to R'$ when $R \to R'$ is not étale. It is obvious that G has good reduction if and only if there exists a Barsotti-Tate group $\mathcal{G}$ over R and an R'-isomorphism $\mathcal{G} \otimes_R R' \simeq \mathcal{G}'$ compatible with the Galois action (using $\widetilde{\phi}_\sigma$'s on $\mathcal{G}'$). In general, given a finite Galois group Γ, we call a pair $(\mathcal{G}', \widetilde{\phi}_\sigma)$ a Γ-*descent datum* where $\mathcal{G}'$ is a finite flat group scheme (a Barsotti-Tate group) over R' and $\widetilde{\phi}_\sigma : \sigma^*(\mathcal{G}') \simeq \mathcal{G}'$ are R'-isomorphisms satisfying the cocycle condition. We say a Γ-descent datum $(\mathcal{G}', \widetilde{\phi}_\sigma)$ has an *effective descent* if there exists a finite flat group scheme (a Barsotti-Tate group) $\mathcal{G}$ over R and an R'-isomorphism $\mathcal{G} \otimes_R R' \simeq \mathcal{G}'$ compatible with the Galois action (using $\widetilde{\phi}_\sigma$'s on $\mathcal{G}'$, see §2 for more details). Using deformation theory of Barsotti-Tate groups, §2 to §4 will prove the following:

Theorem 1.0.3 (Descent Theorem)**.** *The data $(\mathcal{G}', \widetilde{\phi}_\sigma)$ has an effective descent to a Barsotti-Tate group over R if and only if $(\mathcal{G}'[p^{c_0}], \widetilde{\phi}_\sigma)$ has an effective descent to a finite flat group scheme over R, where $c_0 \geq 1$ only depends on e.*

Remark 1.0.4. We will prove the same Theorem in the equi-characteristic case. In such a case, the constant c_0 depends on the *relative* discriminant $\Delta_{K'/K}$ and the height of G. We do not optimize the constant c_0, so there is still room to improve.

The Descent Theorem is the heart part of this paper. Let us sketch the idea of proof as the following: In §2, we develop an *ad hoc* finite Galois descent theory on torsion level (e.g., $R/\mathfrak{m}^n$-level). Using such language, G has good reduction over K if and only if for each $n \geq 1$, the Γ-descent datum $(\mathcal{G}'_n, \widetilde{\phi}_{\sigma,n})$ has an effective descent to $\mathcal{G}_n$ over $R/\mathfrak{m}^n$ and compatible with base change $R'/\mathfrak{m}^{n+1} \to R'/\mathfrak{m}^n$, where $\mathcal{G}'_n = \mathcal{G}' \otimes_{R'} R'/\mathfrak{m}^n$ and $\widetilde{\phi}_{\sigma,n} = \widetilde{\phi}_\sigma \otimes_{R'} R'/\mathfrak{m}^n$. Since R' is totally ramified over R, we see that if such $\mathcal{G}_n$ does exist, $\mathcal{G}_n$ and $\mathcal{G}'_n$ must has same closed fiber $\mathcal{G}_0$. Thus, $\mathcal{G}_n$ can be seen as some "correct" deformations of $\mathcal{G}_0$. Luckily, Messing's theory

provides some crystals to precisely describe deformations of Barsotti-Tate groups. So the proof of Theorem 1.0.3 can be reduced to the effectiveness of descending a short exact sequence of finite free R'-modules with semi-linear Γ-actions coming from G':

$$0 \to M_1' \to M_2' \to M_3' \to 0.$$

In general, unlike the ordinary Galois descent theory, the effective descent of such sequence does not necessarily exist. But work of §2 will prove that such effective descent does exist if for each $i = 1, 2, 3$, $M_i' \otimes_{R'} R'/\mathfrak{m}^d$ has an effective descent to an $R/\mathfrak{m}^d$-module M_i for some constant $d \geq 1$ which depends only on e in the mixed characteristic case and depends only on $\Delta_{K'/K}$ and the height of G in the equi-characteristic case. Finally, in §3, we use deformation theories from Grothendieck and Drinfeld to show that the effective descents of $M_i' \otimes_{R'} R'/\mathfrak{m}^d$ does exist by proving that $\mathcal{G}_d'$ has an effective descent to $\mathcal{G}_d$ under the hypothesis that $\mathcal{G}'[p^{c_0}]$ has an effective descent. Then Theorem 1.0.3 follows.

Note that Theorem 1.0.3 does not immediately prove Theorem 1.1. In fact, although $G[p^{c_0}]$ can be extended to a finite flat group scheme $\mathcal{G}_{c_0}$ over R, $\mathcal{G}_{c_0}$ is not necessarily an effective descent of $\mathcal{G}'[p^{c_0}]$ because the extensions of a finite K-group to a finite flat R-group scheme are not necessarily unique. Hence, in §5, we generalize Tate's isogeny theorem [18] and Raynaud's result §3, [15] to finite level:

Theorem 1.0.5. *Let $\mathcal{G}$ be a truncated Barsotti-Tate group over R of level n, $\mathcal{H}$ a finite flat group scheme over R and $f_K : \mathcal{G}_K \to \mathcal{H}_K$ a K-group scheme morphism. Then there exists a constant $c_1 \geq 2$ depending on the absolute ramification index $e(K/\mathbb{Q}_p)$ and the height of $\mathcal{G}_K$ such that for all $n \geq c_1$ there exists an R-group scheme morphism $F : \mathcal{G} \to \mathcal{H}$ satisfying $F \otimes_R K = p^{c_1} \circ f_K$.*

Proposition 1.0.6. *Let $\mathcal{G}$ be a finite flat group scheme over R whose generic fiber $\mathcal{G}_K$ is a truncated Barsotti-Tate group of level n. There exists a constant $c_2 \geq 2$ depending on e and the height of $\mathcal{G}_K$ such that if $n \geq c_2$ then there exists a truncated Barsotti-Tate group $\mathcal{G}'$ over R of level $n - c_2$ and R-group scheme morphisms $g : \mathcal{G} \to \mathcal{G}'$ and $g' : \mathcal{G}' \to \mathcal{G}$ satisfying*

(1) *$g_K' = g' \otimes_R K$ factors though $\mathcal{G}_K[p^{c_2}]$ and $g_K' : \mathcal{G}_K' \to \mathcal{G}_K[p^{c_2}]$ is an isomorphism*

(2) *$(g \circ g') \otimes_R K = p^{c_2}$*

Combining Theorem 1.0.5 and Proposition 1.0.6, in the beginning of §5, we will see that $c = c_0 + c_1 + c_2$, and if $G[p^c]$ can extend a finite flat group scheme $\mathcal{G}_c$ then there exits a subgroup scheme $\mathcal{G}_{c_0}$ of $\mathcal{G}_c$ such that $\mathcal{G}_{c_0}$ provides an effective descent of $\mathcal{G}'[p^{c_0}]$, hence prove the main Theorem.

Remark 1.0.7. Using the classification of finite flat group schemes over R in §2.3, [12], we can prove that c_1 and c_2, hence c can be chosen to depend only on the absolute ramification index and not on the height. In fact, we can prove Theorem 1.0.5 and Proposition 1.0.6 in the more general setting for *φ-modules of finite height* in the sense of Fontaine ([9]). These results will play a central technical role in our proof of a conjecture of Fontaine ([10]) concerning that the semi-stability for p-adic Galois representations is preserved under suitable inverse limit processes, subject to a natural boundedness hypotheses on the Hodge-Tate weights. The methods of this paper may be viewed as a simpler version of the methods we develop in [13] where we prove Fontaine's conjecture.

When writing of this paper was nearly complete, Bondarko posted his preprint [3] on the `arxiv`, where he gave a new classification of finite flat group schemes over R and obtained a similar result (c.f. Theorem 4.3.2) as our main result, Theorem 1.0.1 above. We first remark that the hypotheses in our Main Theorem are weaker than that of Theorem 4.3.2 in Bondarko's preprint [3]. One of his hypotheses requires the finite flat group scheme which extends the given generic fiber has to have the correct "tangent space". But our hypothesis does not have such a restriction on such finite flat group schemes. Also here we use a totally different method. Bondarko's theory mainly depends on the theory of Cartier modules that gives a classification of formal groups. His method works without a perfectness hypothesis on the residue field and gives slightly better constants (cf. Theorem 4.1.2). But our approach by deformation theory can be adapted to apply to rather general semi-stable p-adic Galois representations without restricting to crystalline representations with Hodge-Tate weights in $\{0, 1\}$ (see [4]). In contrast, Bondarko's method of formal groups seems difficult to apply to such a wider class of representations. To be more precise, the problem we solve in this paper is a special case of the following more general question:

Let T be a $\mathrm{Gal}(\overline{K}/K)$-stable $\mathbb{Z}_p$-lattice in a potentially semi-stable p-adic Galois representation V. Does there exist a constant c such that if $T/p^c T$ is torsion semi-stable then V is semi-stable? If so, how does c depend on the arithmetic of K and V?

Using the machinery provided by ([12]), the author has made some progress on this question by modifying the method in this paper. Also an extended version of Theorem 1.0.5 and Proposition 1.0.6 is used in our proof of a conjecture of Fontaine ([13]), as explained in Remark 1.0.7.

Convention 1.0.8. From §2 to §4, we only assume R is a complete discrete valuation ring with perfect residue field k with $\mathrm{char}(k) = p > 0$, and we select a uniformizer π (resp. π') in the maximal ideal $\mathfrak{m}$ (resp. $\mathfrak{m}'$). In particular, we do not assume that the fraction field K of R has characteristic 0. We fixed a finite Galois extension K'/K with Galois group Γ. We have to consider torsion modules such as $M/\pi^n M$. In order to simplify the notation, we write M/π^n to denote $M/\pi^n M$. We use "Id" to denote the identity map (of modules, schemes, etc.). If M is an R'-module equipped with semi-linear action of Γ, then M^Γ denotes the R-submodule of Γ-invariant elements.

2. Elementary Finite Descent Theory

2.1. **Preliminary.** We will first set up an *ad hoc* "descent theory" from R' to R using Γ-actions. We will see that for "descent data" on a finite free R'-module, the obstruction to effective descent can be determined by working modulo $\mathfrak{m}^n$ for a sufficiently large n depending only on the arithmetic of K'/K.

Definition 2.1.1. Let M', N' be R'-modules. Fix $\sigma \in \Gamma$. A σ-*semi-linear homomorphism* $\phi_\sigma : M' \to N'$ is an R-linear homomorphism such that for all $x \in R'$ and for all $m \in M'$, $\phi_\sigma(xm) = \sigma(x)\phi(m)$. An R'-*semi-linear homomorphism* from

M' to N' is a collection $(\phi_\sigma)_{\sigma \in \Gamma}$ where each ϕ_σ is σ-semi-linear and $\phi_{\sigma\tau} = \phi_\sigma \circ \phi_\tau$ for all $\sigma, \tau \in \Gamma$.

It is obvious that the set of all R'-semi-linear automorphisms of M' forms a group, denoted by $\mathrm{Aut}_\Gamma(M')$

Definition 2.1.2. A Γ-*descent module* structure on an R'-module M' is a pair (M', ϕ), where $\phi : \Gamma \to \mathrm{Aut}_\Gamma(M')$ gives an R'-semi-linear action of Γ on M'.

Let (M', ϕ_σ), (N', ψ_σ) be two Γ-descent modules. A *morphism* $f : (M', \phi_\sigma) \to (N', \psi_\sigma)$ is a R'-linear morphism $f : M' \to N'$ such that for all $\sigma \in \Gamma$, $f \circ \phi_\sigma = \psi_\sigma \circ f$. That is, a morphism is just an R'-module morphism compatible with the Γ-actions. Most of time we use terminology "Galois-equivariantly" instead of "morphism of Γ-descent modules".

Example 2.1.3. Let M be an R-module, so $M' = M \otimes_R R'$ has a natural Γ-descent datum ϕ_σ that comes from the Galois action of Γ on R'. We always omit ϕ_σ when we mention the canonical Γ-descent module $(M \otimes_R R', \phi_\sigma)$.

Definition 2.1.4. A Γ-descent module (M', ϕ_σ) has an *effective descent* if there exists an R-module M such that $M \otimes_R R'$ is isomorphic to (M', ϕ_σ) as Γ-descent modules, or equivalently, if there is a R'-isomorphism $i : M \otimes_R R' \simeq M'$ such that i is Galois equivariant.

Similarly, we can define exact sequences of Γ-descent modules and effective descent for such exact sequences. Also we can generalize such descent language to the setting of schemes. For example, we can define Γ-descent data of an affine scheme (affine group scheme) over R' by defining Γ-descent data of its coordinate ring (Hopf algebra). Similar definitions apply to the notion of effective descent of an affine scheme (affine group scheme) over R'. More precisely, let $X' = \mathrm{Spec}(A')$ where A' is an R'-algebra. It is easy to see that a Γ-descent data structure on A' is equivalent to a collection of R'-isomorphisms $\Phi_\sigma : \sigma^*(X') \simeq X'$ satisfying cocycle conditions $\Phi_{\sigma\tau} = \Phi_\sigma \sigma^*(\Phi_\tau)$ for all $\sigma, \tau \in \Gamma$. Thus we also call such a pair (X', Φ_σ) a Γ-*descent datum*. Let (X'', Φ_σ') be another Γ-descent datum. A *morphism* between Γ-descent data $f : (X', \Phi_\sigma) \to (X'', \Phi_\sigma')$ is morphism of R'-schemes such that the following diagram commutes for all $\sigma \in \Gamma$:

$$(2.1.1) \qquad \begin{array}{ccc} \sigma^*(X') & \xrightarrow{\;\sigma^*(f)\;} & \sigma^*(X'') \\ \Phi_\sigma \downarrow \wr & & \wr \downarrow \Phi_\sigma' \\ X' & \xrightarrow{\quad f \quad} & X'' \end{array}$$

Similarly, we define f to be an *isomorphism* of Γ-descent data if f is an isomorphism of underlying R'-schemes. The inverse is obviously a morphism of Γ-descent data.

Example 2.1.5. Let $X = \mathrm{Spec}(A)$ be an R-scheme. Then we have a canonical Γ-descent data ι_σ on $X \otimes_R R'$ coming from the canonical Γ-descent algebra $A \otimes_R R'$.

We say that a Γ-descent datum (X', Φ_σ) has an *effective descent* if there exists an R-scheme X such that $(X \otimes_R R', \iota_\sigma)$ is isomorphic to (X', Φ_σ) as Γ-descent data. In the affine case, it is equivalent to say that the coordinate ring of X' has an effective descent as a Γ-descent algebra.

Example 2.1.6. As in §1, let G be a Barsotti-Tate group over K which acquire good reduction over K' in the sense that there exists an Barsotti-Tate group $\mathcal{G}'$. By Tate's Theorem [18] (De Jong's Theorem [8] in the equi-characteristic case), we have a Γ-descent datum $(\mathcal{G}', \widetilde{\phi}_\sigma)$ coming from the generic fiber $G \otimes_K K'$. Furthermore, G has good reduction over K if and only if $(\mathcal{G}', \widetilde{\phi}_\sigma)$ has an effective descent and Theorem 1.0.3 claims that such effective descent does exist if and only if $(\mathcal{G}[p^{c_0}], \widetilde{\phi}_\sigma)$ has one for some constant c_0.

Let (X', Φ_σ) and (X'', Φ'_σ) be Γ-descent data and $f : X' \xrightarrow{\sim} X''$ an isomorphism of R'-schemes. The following easy lemma will be very useful to check if f is an isomorphism of Γ-descent data.

Lemma 2.1.7. *Using notations as above, for all $\sigma \in \Gamma$, define*

$$f_\sigma : \ X' \xrightarrow[\sim]{\Phi_\sigma^{-1}} \sigma^*(X') \xrightarrow{\sigma^*(f)} \sigma^*(X'') \xrightarrow[\sim]{\Phi'_\sigma} X'' \ .$$

Then f is an isomorphism of Γ-descent data if and only if $f^{-1} \circ f_\sigma = \mathrm{Id}$ for all $\sigma \in \Gamma$.

Proof. From the diagram (2.1.1), this is clear. $\qquad\square$

2.2. **"Weak" full faithfullness.** Define a functor from the category of R-modules to the category of Γ-descent modules over R':

$$F : M \longmapsto (M \otimes_R R', \phi_\sigma)$$

as in Example 2.1.3. Unlike ordinary descent theory (e.g., *fppf* descent theory), the functor F is neither full nor faithful.

Example 2.2.1. Let $I' \subset R'$ be an ideal not arising from R. The Γ-action on R' preserves I', and $I := I'^\Gamma$ is a nonzero principal ideal of R, but the map $I \otimes_R R' \hookrightarrow I'$ is not surjective. That is, I' with its evident Γ-descent datum does not have an effective descent.

Example 2.2.2. If (M', ϕ_σ) is a Γ-descent module, then $\overline{M'} = M'/\pi^n M'$ has a canonical descent datum $(\overline{M'}, \bar{\phi}_\sigma)$. We still use ϕ_σ to denote the semi-linear Galois action on $\overline{M'}$. Similarly, if (X, Φ_σ) is a Γ-descent data, we have natural Γ-descent data $\Phi_\sigma \otimes_R R/\pi^n$ on $X \otimes_R R/\pi^n$.

Example 2.2.3. Let $M' = R'/\pi^n$. There is a natural descent datum ϕ_σ on M' from the Galois action of Γ on R'. We see that $R/\pi^n \to (R'/\pi^n)^\Gamma$ is injective. However in general, the equality does not necessary hold (In fact, $\mathrm{H}^1(\Gamma, R')[\pi^n]$ is not necessarily trivial, especially when $p | e(K'/K)$). Select an $\alpha \in (R'/\pi^n)^\Gamma - R/\pi^n$ such that $R'/\pi^n = (R'/\pi^n) \cdot \alpha$. Let $M_1 = R/\pi^n$ and $M_2 = (R/\pi^n) \cdot \alpha$, so M_1 and M_2 are two effective descents of (M', ϕ_σ). Note that the identity $\mathrm{Id} : M' \to M'$ is a morphism between these two descent data on M', but there does not exist a morphism $f : M_1 \to M_2$ such that $f \otimes_R R' = \mathrm{Id}$.

As we have seen above, the functor F is not a fully faithful functor. However, we will show F actually has "weak" full faithfulness in the following sense.

Proposition 2.2.4. *Let $f : (M'_1, \phi_1) \to (M'_2, \phi_2)$ be a morphism of Γ-descent modules with M'_1, M'_2 finite free R'/π^n-modules. Suppose M_1, M_2 are effective descents for M'_1, M'_2 respectively. Then M_1 and M_2 are finite free R/π^n-modules.*

Furthermore, there exists a constant r depending on R' and R such that for all $n \geq r$ the morphism $\pi^r f$ has a unique effective descent to a morphism $\hat{f} : M_1 \to M_2$.

In Lemma 2.2.9 below, we will see how the constant r depends on the arithmetic of R' and R.

Corollary 2.2.5. *If f in Proposition 2.2.4 is an isomorphism, then the natural map $\bar{f} : M_1/\pi^{n-r} \to M_2/\pi^{n-r}$ induced by $\hat{f}$ is an isomorphism.*

Corollary 2.2.6. *With notations as above, if M_1' is a finite free R'-module then the effective descent of (M_1, ϕ_1) is unique up to the unique isomorphism if it exists. Moreover, the formation of such descent (when it exists) is uniquely effective on the level of morphisms.*

Remark 2.2.7. Of course, Corollary 2.2.6 is obvious from considering $M_1' \otimes_{R'} K'$ over K'. However for our purposes what matters is that such an effective descent can be studied using the effective descent at torsion level.

Let $S' : 0 \to (M_1', \phi_1) \to (M_2', \phi_2) \to (M_3', \phi_3) \to 0$ be an exact sequence of Γ-descent modules. Recall that we say that S' has an effective descent if there exists an *exact* sequence of R-modules $S : 0 \to M_1 \to M_2 \to M_3 \to 0$ such that the following diagram commutes.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_1' & \longrightarrow & M_2' & \longrightarrow & M_3' & \longrightarrow & 0 \\
 & & f_1 \uparrow \wr & & f_2 \uparrow \wr & & f_3 \uparrow \wr & & \\
0 & \longrightarrow & M_1 \otimes_R R' & \longrightarrow & M_2 \otimes_R R' & \longrightarrow & M_3 \otimes_R R' & \longrightarrow & 0
\end{array}
$$

where f_i is a Galois-equivariant isomorphism for $i = 1, 2, 3$. Using Corollary 2.2.6, we get:

Corollary 2.2.8. *Suppose M_i' is a finite free R'-module for each $i = 1, 2, 3$. Then S' has an effective descent if and only if (M_i', ϕ_i) has an effective descent for $i = 1, 2, 3$.*

Before proving Proposition 2.2.4, we need a lemma to bound Galois cohomology and precisely describe the constant r in Proposition 2.2.4. Let $\Delta_{K'/K}$ be the discriminant of R'/R, r_0 the p-index of Γ (that is $p^{r_0} \parallel \#(\Gamma)$). We normalize the valuation $v(\cdot)$ on R' by putting $v(\pi) = 1$, so $v(K'^{\times}) = e(K'/K)^{-1} \cdot \mathbb{Z}$ where $e(K'/K)$ is the relative ramification index. Finally, we put $r = v(\Delta_{K'/K})$ in the equi-characteristic case and $r = r_0 v(p)$ in the mixed characteristic case.

Lemma 2.2.9. *Let (M', ϕ_σ) be a Γ-descent module with M' a finite free R'-module of rank m. The element π^r kills $\mathrm{H}^1(\Gamma, M')$ in the mixed characteristic case. In the equi-characteristic case, $\mathrm{H}^1(\Gamma, M') = 0$ if $r_0 = 0$ and $\pi^{m(r+1)}$ kills $\mathrm{H}^1(\Gamma, M')$ if $r_0 > 0$.*

Proof. By Hilbert's Theorem 90, $\mathrm{H}^1(\Gamma, M' \otimes_{R'} K') = 0$. Thus, there exists a positive integer $\tilde{r} \geq 0$ such that $\pi^{\tilde{r}}$ kills the finite R-module $\mathrm{H}^1(\Gamma, M')$. Hence, in both cases (equi-characteristic and mixed characteristic) there exists some $\hat{r} \geq 1$ such that $p^{\hat{r}}$ kills $\mathrm{H}^1(\Gamma, M')$. On the other hand, note that $\#(\Gamma)$ (the order of Γ) also kills $\mathrm{H}^1(\Gamma, M')$. Thus, in the mixed characteristic case, $p^{r_0} \in (\pi)^r R^\times$ kills $\mathrm{H}^1(\Gamma, M')$.

Now let us deal with the equi-characteristic case, which needs a much more complicated argument. We begin by treating the case $M' = R'$. First let us reduce the problem to the case that Γ is cyclic with order p.

Note that if $\#(\Gamma)$ is prime to p, since p also kills $H^1(\Gamma, R')$ we have $H^1(\Gamma, R') = 0$. On the other hand, since Γ is solvable, by the inflation-restriction exact sequence

$$(2.2.1) \qquad 0 \longrightarrow H^1(\Gamma/\Gamma', R'^{\Gamma'}) \xrightarrow{\text{Inf}} H^1(\Gamma, R') \xrightarrow{\text{Res}} H^1(\Gamma', R'),$$

where Γ' is the wild inertia subgroup of Γ, we reduce the problem to the case that the order of Γ is a power of p.

Now we can assume Γ' and Γ in (2.2.1) are of p-power order and we assume that the lemma is proven for cyclic groups of order p. We assume by induction on $\#(\Gamma)$ that it is settled for smaller p-power order and we take $\Gamma' \triangleleft \Gamma$ a nontrivial normal subgroup. Let K_1 be the fraction field of $R'^{\Gamma'}$ and π_1 a uniformizor $R'^{\Gamma'}$. By induction, $\pi^{v(\Delta_{K_1/K})}$ kills $H^1(\Gamma/\Gamma', R'^{\Gamma'})$ and $\pi_1^{v_1(\Delta_{K'/K_1})}$ kills $H^1(\Gamma', R')$, where v_1 is the valuation on R' by putting $v_1(\pi_1) = 1$. By the transitivity formula for discriminants,

$$\pi^{v(\Delta_{K_1/K})} \cdot N_{K_1/K}(\pi_1^{v_1(\Delta_{K'/K_1})}) = u\pi^{v(\Delta_{K'/K})}$$

where u is a unit in R'. This product kills $H^1(\Gamma, R')$. Thus, it remains to prove that $\pi^{v(\Delta_{K'/K})}$ kills $H^1(\Gamma, R')$ when Γ is cyclic order p.

Now suppose that Γ is cyclic with order p. Let φ be a cocycle in $H^1(\Gamma, R')$ and σ a generator of Γ. Since $H^1(\Gamma, K') = 0$, we have $\varphi(\sigma) = \sigma(x/\pi^n) - x/\pi^n$ for an $x \in R'$ and some $n \geq 0$. Write $x = \sum_{i=0}^{p-1} x_i(\pi')^i$, where $x_i \in R$ for $i = 0, \ldots, p-1$. Then $\varphi(\sigma) = \frac{1}{\pi^n} \sum_{i=1}^{p-1} x_i(\sigma(\pi')^i - (\pi')^i)$. Note that $\varphi(\sigma) \in R'$, so we have

$$(2.2.2) \qquad \sum_{i=1}^{p-1} x_i(\sigma(\pi')^i - (\pi')^i) \equiv 0 \mod \pi^n$$

Recall that $r = v(\Delta_{K'/K})$ by definition. Now it suffices to prove that if $n > r$ then $x_i \equiv 0 \mod \pi^{n-r}$ for all $i = 1, \ldots, p-1$.

Since Γ is wild inertia, there exists a $\tau \in \Gamma$ with $\tau \neq \text{Id}$ such that

$$\tau(\pi') - \pi' \equiv 0 \mod (\pi')^2$$

From (2.2.2) we have

$$\left(\sum_{i=1}^{p-1} x_i \left(\frac{\tau(\pi')^i - \pi'^i}{\tau(\pi') - \pi'} \right) \right) (\tau(\pi') - \pi') \equiv 0 \mod \pi^n$$

Since $v(\tau(\pi') - \pi') \leq v(\Delta_{K'/K}) = r$,

$$(2.2.3) \qquad \sum_{i=1}^{p-1} x_i \left(\frac{\tau(\pi')^i - \pi'^i}{\tau(\pi') - \pi'} \right) \equiv 0 \mod \pi^{n-r}.$$

To prove that $x_i \equiv 0 \mod \pi^{n-r}$, $i = 1, \ldots, p-1$, it suffices (by induction on $n > r$) to prove that $\pi | x_i$ for all $i = 1, \ldots, p-1$. First from (2.2.3), we see that $\pi' | x_1$, so $\pi | x_1$. Suppose j is the minimal such that $\pi \nmid x_j$, or equivalently $\pi' \nmid x_j$. We prove that this is impossible by the following claim:

Claim:
$$v(\frac{\tau(\pi')^i - \pi'^i}{\tau(\pi') - \pi'}) = \frac{i-1}{p}, \; i = 1, \ldots, p-1.$$

Assuming the claim is true for a moment, we have
$$v\left(\sum_{i=1}^{p-1} x_i(\frac{\tau(\pi')^i - \pi'^i}{\tau(\pi') - \pi'})\right) = v(x_j(\frac{\tau(\pi')^j - \pi'^j}{\tau(\pi') - \pi'})) = \frac{j-1}{p} < 1$$

From (2.2.3), we see this is impossible.

Finally, let us compute $v(\frac{\tau(\pi')^i - \pi'^i}{\tau(\pi') - \pi'})$ to prove the Claim. By the binomial expansion,

$$\tau(\pi')^i - \pi'^i = (\pi' + (\tau(\pi') - \pi'))^i - \pi'^i = \sum_{m=1}^{i} \binom{i}{m}(\pi')^{i-m}(\tau(\pi') - \pi')^m.$$

Since $v(\tau(\pi') - \pi') \geq v(\pi')^2$, we have $v(\frac{\tau(\pi')^i - \pi'^i}{\tau(\pi') - \pi'}) = v(i\pi'^{i-1}) = \frac{i-1}{p}$. This proves the Claim.

Now let us treat the general case for M' a finite free R'-module rank m. The case $r_0 = 0$ is clear since $\#(\Gamma) \in R^\times$ in such case. Now consider the case $r_0 > 0$. Let us first reduce the problem to the case that M' is free R'-rank 1. In fact, we have Galois descent data on $M' \otimes_{R'} K'$ from ϕ_σ. By ordinary Galois descent, $M' \otimes_{R'} K'$ has an effective descent. Thus we can find a nonzero $\alpha \in M'$ such that $\phi_\sigma(\alpha) = \alpha$. Let M'_1 be the saturation of $R' \cdot \alpha \subset M'$; that is, M'_1 is the smallest submodule of M' containing $R' \cdot \alpha$ such that M'/M'_1 is a free R'-module. It is easy to see that M'_1 is a Galois-stable submodule of M'. Thus, we have an exact sequence of a Γ-descent modules,
$$0 \to M'_1 \to M' \to M'/M'_1 \to 0,$$
where M'/M'_1 is a finite free Γ-descent module with R'-rank $m-1$. By induction on m, it now suffices to prove the lemma for the case when M' is free R'-module with rank 1.

Now assume M' is free rank-1 R'-module. Select a basis $\alpha \in M'$. The semi-linear Galois structure on M' is determined by the 1-cocycle $\sigma \mapsto \phi_\sigma(\alpha)/\alpha \in R'^\times$. By Hilbert's Theorem 90, we see there exists an $x \in K'^\times$ such that $\sigma(x)/x = \phi_\sigma(\alpha)/\alpha$ for all $\sigma \in \Gamma$. After multiplying some power of π, we can assume $0 \leq v(x) < 1$. Define an R'-module morphism $f_x : M' \to R' \cdot x \subset R'$ by $\alpha \mapsto x$. Since $\sigma(x)/x = \phi_\sigma(\alpha)/\alpha$, we see that f_x is Galois-equivariant. From the short exact sequence

$$0 \longrightarrow M' \xrightarrow{f_x} R' \longrightarrow R'/f_x(M') \longrightarrow 0,$$

we get a short exact sequence $(R'/f_x(M'))^\Gamma \to \mathrm{H}^1(\Gamma, M') \to \mathrm{H}^1(\Gamma, R')$. Note that $f_x(M') = xR'$ and $0 \leq v(x) < 1$, so π kills $(R'/f_x(M'))^\Gamma$. As we have seen above, π^r kills $\mathrm{H}^1(\Gamma, R')$, so π^{r+1} kills $\mathrm{H}^1(\Gamma, M')$.
$$\square$$

Remark 2.2.10. Of course, the above proof for the equi-characteristic case also works for the mixed-characteristic case. Unfortunately, $v(\Delta_{K'/K})$ is always larger than $v(p)r_0$ in this case, so we use $v(p)r_0$ instead of $v(\Delta_{K'/K})$. This suggests that there may still be room to improve the constant r.

Proof of Proposition 2.2.4. Since R'/π^n is faithfully flat over R/π^n and M_i' is a finite free R'/π^n-module, M_i is a finite flat (hence free)R/π^n-module.

Now consider the following commutative diagram:

$$
\begin{array}{ccccc}
M_1 & \overset{i}{\hookrightarrow} & (M_1')^\Gamma & \overset{f^\Gamma}{\longrightarrow} & (M_2')^\Gamma \\
& & \downarrow{\scriptstyle \mathrm{pr}_1^\Gamma} & & \downarrow{\scriptstyle \mathrm{pr}_2^\Gamma} \\
& & (M_1')^\Gamma & \overset{f^\Gamma}{\longrightarrow} & (M_2')^\Gamma
\end{array}
$$

where pr_1^Γ, pr_2^Γ are induced by the morphism π^r and i is the natural embedding. Since M_i' is a finite free R'/π^n-module, the morphism $f^\Gamma \circ \mathrm{pr}_1^\Gamma \circ i$ is an effective descent of $\pi^r f$ by the following lemma. The uniqueness of this descent is clear because R'/π^n is flat over R/π^n. $\qquad\square$

Lemma 2.2.11. *Let* $\mathrm{pr}^\Gamma : (R'/\pi^n)^\Gamma \to (R'/\pi^n)^\Gamma$ *be the morphism induced by* π^r. *The image of* pr^Γ *is* $\pi^r R/\pi^n$.

Proof. Consider the following commutative diagram of exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & R'^\Gamma & \overset{\pi^n}{\longrightarrow} & R'^\Gamma & \longrightarrow & (R'/\pi^n R')^\Gamma & \longrightarrow & \mathrm{H}^1(\Gamma, R')[\pi^n] & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \pi^r} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \pi^r} \\
0 & \longrightarrow & R'^\Gamma & \overset{\pi^{n-r}}{\longrightarrow} & R'^\Gamma & \overset{j}{\longrightarrow} & (R'/\pi^{n-r} R')^\Gamma & \longrightarrow & \mathrm{H}^1(\Gamma, R')[\pi^{n-r}] & \longrightarrow & 0
\end{array}
$$

where $\mathrm{H}^1(\Gamma, R')[\pi^n]$ is the kernel of π^n on $\mathrm{H}^1(\Gamma, R')$ and h is induced by the canonical reduction map. We need to show h has image $R/\pi^{n-r} = \mathrm{Im}(j)$. By Lemma 2.2.9, π^r kills $\mathrm{H}^1(\Gamma, R')$, so $\mathrm{Im}(\mathrm{pr}^\Gamma) = \mathrm{Im}(j) = (R'^\Gamma)/\pi^{n-r} = R/\pi^{n-r}$. $\qquad\square$

Convention 2.2.12. Unfortunately, from now on we cannot deal with the mixed characteristic case and equi-characteristic case in the same relative setting. More precisely, in the mixed characteristic case we can only state and prove results for "p^n-torsion" instead of "π^n-torsion" as in the equi-characteristic case, although the results and ideas of proofs are almost the same if we replace "p" by "π". To save space, we will state the results in mixed characteristic case and use "(resp.)" to indicate the respective results in equi-characteristic.

2.3. Criterion for effectiveness of descent. Now we will show that for a given Galois group, the effectiveness of a finite free Γ-descent R'-module (M', ϕ_σ) will be determined by working at a large torsion level determined by R' and R. Recall that r_0 denotes the p-index of Γ (i.e., $p^{r_0} \parallel \#(\Gamma)$). Let $d = \mathrm{Max}(1, 2r_0)$ in the mixed characteristic case. In the equi-characteristic case, let $d = 1$ if $r_0 = 0$ and $d = 2m(r+1)$ if $r_0 \geq 1$, where m is the R'-rank of M'.

Proposition 2.3.1. *Let* (M', ϕ_σ) *be a* Γ-*descent module with* M' *a finite free* R'-*module of rank* m. *Then* (M', ϕ_σ) *has an effective descent if and only if* $(M'/p^d, \phi_\sigma)$ *(resp.* $(M'/\pi^d, \phi_\sigma)$ *) has an effective descent to an* R/p^d-*(resp.* R/π^d-*)module (necessarily finite and free).*

Proof. The "only if" direction is trivial. For the converse, let us first prove the proposition in the mixed characteristic case. First consider the case $d = 2r_0$. Note

$r_0 = r/v(p)$ in this case, so p^{r_0} kills $\mathrm{H}^1(\Gamma, M')$ by Lemma 2.2.9. Consider following diagram:

(2.3.1)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{p^{r_0}} & M' & \longrightarrow & M'/p^{r_0} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M'^{\Gamma} & \xrightarrow{p^{r_0}} & M'^{\Gamma} & \xrightarrow{\alpha} & (M'/p^{r_0})^{\Gamma} & \xrightarrow{\beta} & \mathrm{H}^1(\Gamma, M')[p^{r_0}] \longrightarrow 0
\end{array}
$$

where $\mathrm{H}^1(\Gamma, M')[p^{r_0}]$ is the kernel of p^{r_0}. Let $M = M'^{\Gamma}$ and let $\mathrm{Im}(\alpha)$ denote the image of map α. By Galois descent theory for K'/K, M is a finite free R-module with the same rank as that of M' over R'. Obviously, $\alpha(M) \simeq M/p^{r_0}M$.

We now claim that $\mathrm{Im}(\alpha)$ is an effective descent of M'/p^{r_0}; that is, the natural Γ-equivariant map $\alpha(M) \otimes_R R' \to M'/p^{r_0}$ is an isomorphism.

To prove this claim, let us consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{p^{2r_0}} & M' & \longrightarrow & M'/p^{2r_0}M' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p^{r_0}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & M' & \xrightarrow{p^{r_0}} & M' & \longrightarrow & M'/p^{r_0}M' & \longrightarrow & 0
\end{array}
$$

Take Galois invariants to get a new diagram, with M denoting M'^{Γ}:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{p^{2r_0}} & M & \longrightarrow & (M'/p^{2r_0}M')^{\Gamma} & \longrightarrow & \mathrm{H}^1(\Gamma, M')[p^{2r_0}] \longrightarrow 0 \\
 & & \downarrow{\scriptstyle p^{r_0}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle i^{\Gamma}} & & \downarrow{\scriptstyle p^{r_0}} \\
0 & \longrightarrow & M & \xrightarrow{p^{r_0}} & M & \xrightarrow{\alpha} & (M'/p^{r_0}M')^{\Gamma} & \xrightarrow{\beta} & \mathrm{H}^1(\Gamma, M')[p^{r_0}] \longrightarrow 0
\end{array}
$$

Since p^{r_0} kills $\mathrm{H}^1(\Gamma, M')$, the right square of the above diagram implies $\mathrm{Im}(i^{\Gamma}) \subseteq \mathrm{Ker}(\beta)$ and the middle square gives us $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(i^{\Gamma})$. By the exactness of α and β, we have $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$. Thus $\mathrm{Im}(\alpha) = \mathrm{Im}(i^{\Gamma})$.

By hypothesis, since $2r_0 = d$, $M'/p^{2r_0}M'$ has an effective descent to an R'/p^{2r_0}-module which is necessarily flat. Hence, $M'/p^{2r_0}M'$ is spanned by $(M'/p^{2r_0}M')^{\Gamma}$ over R'. Thus, $M'/p^{r_0}M'$ is spanned over R' by the image of $(M'/p^{2r_0}M')^{\Gamma}$ in $(M'/p^{r_0}M')^{\Gamma}$, hence by $\mathrm{Im}(\alpha) \simeq M/p^{r_0}M$. Thus, the natural map $\alpha(M) \otimes_R R' \to M'/p^{r_0}M'$ is surjective. Both sides are finite free R'/p^{r_0}-modules with the *same* R'/p^{r_0}-rank, so this natural map is an isomorphism. This completes the proof that $\alpha(M)$ is an effective descent of $M'/p^{r_0}M'$.

Now let us consider the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{p^{r_0}} & M' & \longrightarrow & M'/p^{r_0}M' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle g} & & \\
0 & \longrightarrow & M \otimes_R R' & \xrightarrow{p^{r_0}} & M \otimes_R R' & \longrightarrow & (M/p^{r_0}M) \otimes_R R' & \longrightarrow & 0
\end{array}
$$

where $M = M'^{\Gamma}$ and h, g are the natural maps. We have proved that the natural map g is an isomorphism, so by Nakayama's lemma h is surjective. Since M is

a finite free R-module with R-rank equal to the R'-rank of M', h must be an isomorphism. This settles the proof when $d = 2r_0$ in the mixed characteristic case.

In the mixed characteristic case with $d = 1$, since $\mathrm{H}^1(\Gamma, M') = 0$ the usual projection argument shows that $M/p \simeq (M'/p)^\Gamma$. The hypothesis on M'/p having an effective descent implies that $(M'/p)^\Gamma \otimes_{R/p} R'/p \to M'/p$ is surjective, so $R' \otimes_R M \to M'$ induces a surjective map modulo p and hence is surjective. This surjection must be an isomorphism for freeness reason, as above.

Using Lemma 2.2.9, the proof for the equi-characteristic case is the same as above if we replace p by π and r_0 by $m(r + 1)$ everywhere. $\qquad\square$

3. The Deformation Lemma

We will use results of Grothendieck and Drinfeld to show that under the hypothesis of the Theorem 1.0.3, the effective descent for $(\mathcal{G}', \widetilde{\phi}_\sigma)$ does exist at some R/π^n-level (see Theorem 3.2.1 below). First we need to recall several facts concerning deformations of *truncated Barsotti-Tate groups*.

3.1. Deformations theory from Grothendieck and Drinfeld.

Recall that a *truncated Barsotti-Tate group of level $N \geq 2$ with height h* over a scheme S is an inductive system $(G_n, i_n)_{0 \leq n \leq N}$ of finite locally free commutative S-group schemes and closed immersions $i_n : G_n \to G_{n+1}$ such that

   (1) $|G_n| = p^{nh}$
   (2) i_n identifies G_n with $G_{n+1}[p^n]$ for $0 \leq n < N$.

A truncated Barsotti-Tate group of level 1 with height h is finite locally free S-group scheme killed by p with order p^h such that the sequence

$$G_0 \xrightarrow{\ F\ } G_0^{(p)} \xrightarrow{\ V\ } G_0$$

is exact over $\mathrm{Spec}(\mathcal{O}_S/p\mathcal{O}_S)$, where F is the relative "Frobenius" and V is "Verschiebung" and $G_0 = G \times_S \mathrm{Spec}(\mathcal{O}_S/p\mathcal{O}_S)$.

Now let us recall a result of Grothendieck on truncated Barsotti-Tate groups.

Theorem 3.1.1 (Grothendieck). *Let $i : S \to S'$ be a nilimmersion of schemes and G a truncated Barsotti-Tate group over S of level $n \geq 1$. Suppose S' is affine.*

   (1) *There exists a truncated Barsotti-Tate group G'_n over S' of level n such that G'_n is a deformation of G_n.*
   (2) *If S is complete local noetherian ring with perfect residue field of characteristic p, then there exists a Barsotti-Tate group H such that $G = H[p^n]$.*
   (3) *If there exists a Barsotti-Tate group H over S such that $G = H[p^n]$, then for any deformation of G' of G over S', there exists a Barsotti-Tate group H' over S' such that H' is a deformation of H and $G' \simeq H'[p^n]$ (as a deformation of G).*

Proof. See pp. 171-178, [17]. $\qquad\square$

We next review Drinfeld's results (§1, [11]) on deformations of Barsotti-Tate groups. Let A be a ring, $N \geq 1$ an integer such that N kills A, and I an ideal in A satisfying $I^{\mu+1} = 0$ for a fixed $\mu > 1$. Let $A_0 = A/I$.

Lemma 3.1.2 (Drinfeld). *Let G and H be Barsotti-Tate groups over A.*

(1) *The groups* $\mathrm{Hom}_{A\text{-gp}}(G, H)$ *and* $\mathrm{Hom}_{A_0\text{-gp}}(G_0, H_0)$ *have no non-trivial N-torsion.*

(2) *The natural map* $\mathrm{Hom}_{A\text{-gp}}(G, H) \to \mathrm{Hom}_{A_0\text{-gp}}(G_0, H_0)$ *is injective.*

(3) *For any homomorphism $f_0 : G_0 \to H_0$, there is a unique homomorphism "$N^\mu f$" : $G \to H$ which lifts $N^\mu f_0$.*

(4) *In order that a homomorphism $f_0 : G_0 \to H_0$ lift to a (necessarily unique) homomorphism $f : G \to H$, it is necessary and sufficient that the homomorphism "$N^\mu f$" : $G \to H$ annihilate the subgroup $G[N^\mu]$ of G.*

Proof. See §1, [11]. ☐

Remark 3.1.3. Drinfeld actually shows the above results for more general *fppf group sheaves*, including Barsotti-Tate groups as a special case.

Now let us refine Lemma 3.1.2 for our purposes.

Lemma 3.1.4. *Keep the notations in Lemma* 3.1.2 *with* $N = p^m$ *and* $\mu = 1$. *Choose an integer* $l \geq 1$. *Let* $f_0 : G_0 \to H_0$ *be a morphism of Barsotti-Tate groups over* A_0. *If there exists an A-group morphism* $g : G[p^{m+l}] \to H[p^{m+l}]$ *such that* $g \otimes_A A_0 = f_0|_{G_0[p^{m+l}]}$, *then there exists an A-morphism* $f : G \to H$ *such that* $f \otimes_A A_0 = f_0$ *and* $f|_{G[p^l]} = g|_{G[p^l]}$. *Furthermore, if g and f_0 are isomorphisms, then f is an isomorphism.*

Proof. Using Lemma 3.1.2 in our case, we first prove we can lift the morphism f_0 to an A-morphism. By Lemma 3.1.2 (4), it is necessary that the lift of "$p^m f_0$" annihilates the subgroup $G[p^m]$. Let B be any A-algebra, $B_0 = B/IB$ and $F =$ "$p^m f_0$". By §1, [11], F is defined in the following (using formal smoothness of H):

$$(3.1.1) \quad \begin{array}{ccc} G(B) & \xrightarrow{\quad F \quad} & H(B) \\ & \searrow & \nearrow {\scriptstyle p^m \times \text{any lifting}} \\ & G_0(B_0) \xrightarrow{f_0} H_0(B_0) & \end{array}$$

Since $g \otimes_A A_0 = f_0|_{G_0[p^{m+l}]}$, we have the following commutative diagram:

$$(3.1.2) \quad \begin{array}{ccc} G[p^{m+l}](B) & \xrightarrow{\quad g \quad} & H[p^{m+l}](B) \\ \downarrow & & \downarrow \\ G_0[p^{m+l}](B_0) & \xrightarrow{f_0} & H_0[p^{m+l}](B_0) \end{array}$$

where the columns are reduction modulo I. Now for all $x \in G[p^m](B)$, $g(x)$ is a lift of $f_0(x \mod I)$. Note that $g(x) \in H[p^m](B)$, so $F(x) = p^m(g(x)) = 0$. Thus, $F = $ "$p^m f_0$" kills $G[p^m]$, so we can define $f : G \simeq G/G[p^m] \to H$ induced by F and thus $f : G \to H$ lifts f_0 and $p^m f = F$. Tracing the definition of F in (3.1.1) and (3.1.2), we get $F|_{G[p^{m+l}]} = p^m g$. Thus $f, g : G[p^l] \rightrightarrows H[p^l]$ have the same composite with the *fppf* map $p^m : G[p^{m+l}] \to G[p^l]$, so $f|_{G[p^l]} = g|_{G[p^l]}$.

If f_0 and g are isomorphisms, we can use f_0^{-1} and g^{-1} to construct an A-morphism of Barsotti-Tate groups $\tilde{f} : H \to G$ which lifts f_0^{-1}. We see that $(f \circ \tilde{f}) \otimes_A A_0$ and $(\tilde{f} \circ f) \otimes_A A_0$ are identity maps. By Lemma 3.1.2 (4), $f \circ \tilde{f}$ and $\tilde{f} \circ f$ have to be identity maps, so f is an isomorphism.

☐

3.2. **Application to descent.** We use notations as in Theorem 1.0.3, so $(\mathcal{G}', \widetilde{\phi}_\sigma)$ is an R'-Barsotti-Tate group equipped with Γ-descent datum constructed from $G \otimes_K K'$ over K' as in Introduction. As in Convention 2.2.12, we will state results in the mixed characteristic case and use "(resp.)" to indicate the corresponding results in the equi-characteristic case. Let $\lambda = 1$ for $p \neq 2$ and $\lambda = 2$ for $p = 2$. For any real number x, define $\langle x \rangle$ the maximal integer $< x$, so $\langle x \rangle = x - 1$ if $x \in \mathbb{Z}$. Put $\nu = \langle \log_2 e \rangle + \lambda$ in the mixed characteristic case and $\nu = \langle \log_2 \hat{e} \rangle + \lambda$ in the equi-characteristic case, where $\hat{e}$ is the relative ramification index $e(K'/K)$. For each $n \geq 1$, let $\mathcal{G}'_n = \mathcal{G}' \otimes_{R'} R'/p^n$ and $l_n = 3n + \nu$ in the mixed characteristic case, and $\mathcal{G}'_n = \mathcal{G}' \otimes_{R'} R'/\pi^n$ and $l_n = \langle \log_2 n \rangle + 1 + \nu$ in the equi-characteristic case. We will use Grothendieck's theorem and the same argument of Drinfeld to prove the following:

Theorem 3.2.1 (Deformation Lemma)**.** *If $\mathcal{G}'[p^{l_n}]$ has an effective descent to a truncated Barsotti-Tate group $\mathcal{H}$ over R, then there exists a Barsotti-Tate group $\mathcal{G}_n$ over R/p^n (resp. R/π^n) such that $\mathcal{G}_n$ is an effective descent for $\mathcal{G}'_n$.*

Remark 3.2.2. We do not claim to identify with $\mathcal{H}$ with $\mathcal{G}_n[p^{l_n}]$ as descents of $\mathcal{G}'_n[p^{l_n}]$.

Proof. Let $\mathcal{G}'_0 = \mathcal{G}' \otimes_{R'} k$ be the closed fiber of $\mathcal{G}'$ over k. Since R and R' have the same residue field, it is obvious that the closed fibre $\mathcal{H}_0 = \mathcal{H} \otimes_R k$ is naturally isomorphic to $\mathcal{G}'_0[p^{l_n}]$. By Theorem 3.1.1, there exists a Barsotti-Tate group $\mathcal{G}_n$ over R/p^n (resp. R/π^n) such that $\mathcal{G}_n$ is a deformation of $\mathcal{G}'_0$ over R/p^n (resp. R/π^n) and $\mathcal{H}_n = \mathcal{G}_n[p^{l_n}]$ as a deformation of $\mathcal{H}_0 \simeq \mathcal{G}'_0[p^{l_n}]$, where $\mathcal{H}_n = \mathcal{H} \otimes_R R/p^n$ (resp. $\mathcal{H}_n = \mathcal{H} \otimes_R R/\pi^n$). It now suffices to prove:

Lemma 3.2.3. *There exists an Γ-equivariant isomorphism $f : \mathcal{G}'_n \xrightarrow{\sim} \mathcal{G}_n \otimes_R R'$ as Barsotti-Tate groups over R' with f lifting the identification of $\mathcal{G}'_0$ with the closed fiber $\mathcal{G}_0$ of $\mathcal{G}_n$.*

For the proof of Lemma 3.2.3, first note that both $\mathcal{G}'_n$ and $\mathcal{G}_n \otimes_R R'$ are deformations of $\mathcal{G}'_0$ over R'/p^n (resp. R'/π^n). Since $\mathcal{H}_n$ is an effective descent of $\mathcal{G}'_n[p^{l_n}]$ and $\mathcal{H}_n = \mathcal{G}_n[p^{l_n}]$ as deformations of $\mathcal{H}_0 \simeq \mathcal{G}_0[p^{l_n}]$, there exists a Galois-equivariant isomorphism $g : \mathcal{G}'_n[p^{l_n}] \xrightarrow{\sim} (\mathcal{G}_n \otimes_R R')[p^{l_n}]$ such that the isomorphism $f_0 : \mathcal{G}'_0 \xrightarrow{\sim} \mathcal{G}_n \otimes_R k$ agrees with g on p^{l_n}-torsion over k.

Now let us first discuss the proof for the case that R has mixed characteristic. Recall that $l_n = 3n + \nu = 3n + \langle \log_2 e \rangle + \lambda$ in this case. Consider the following successive square-zero thickenings, where $\mathfrak{m}'$ is the maximal ideal of R':

$$k = R'/\mathfrak{m}' \leftarrow R'/\mathfrak{m}'^2 \leftarrow \cdots \leftarrow R'/\mathfrak{m}'^{2^s} \leftarrow R'/\mathfrak{m}'^{2^{s+1}} \leftarrow \cdots \leftarrow R'/\mathfrak{m}'^{2^{\langle \log_2 e \rangle}} \leftarrow R'/pR'$$

By Lemma 3.1.4 and induction on $1 \leq s \leq \langle \log_2 e \rangle$ with $N = p$, there exists an isomorphism $f_1 : \mathcal{G}'_1 \xrightarrow{\sim} \mathcal{G}_1 \otimes_{R/p} R'/p$ such that $f_1|_{\mathcal{G}'_1[p^{3n+\lambda-1}]} = (g \otimes_{R'/p^n} R'/p)|_{\mathcal{G}'_1[p^{3n+\lambda-1}]}$.

Consider deformations of $\mathcal{G}'_1 = \mathcal{G}'_n \otimes_{R'/p^n} R'/p$ along the following:

$$R'/p \leftarrow R'/p^2 \leftarrow \cdots \leftarrow R'/p^{2^s} \leftarrow R'/p^{2^{s+1}} \leftarrow \cdots \leftarrow R'/p^{2^{\langle \log_2 n \rangle}} \leftarrow R'/p^n.$$

By induction on $1 \leq s \leq \langle \log_2 n \rangle$ and Lemma 3.1.4 with $N = p^{2^s}$ ($N = p^n$ in the last step), there exists an isomorphism $f_n : \mathcal{G}'_n \xrightarrow{\sim} \mathcal{G}_n \otimes_{R/p^n} R'/p^n$ such that

$f_n|_{\mathcal{G}'_n[p^{3n-1+\lambda-t}]} = g|_{\mathcal{G}'_n[p^{3n-1+\lambda-t}]}$, where $t = (\sum_{i=1}^{\langle \log_2 n \rangle} 2^i) + n = 2^{\langle \log_2 n \rangle + 1} - 2 + n \le 3n - 1$. Thus we have $f_n|_{\mathcal{G}'_n[p^\lambda]} = g|_{\mathcal{G}'_n[p^\lambda]}$.

In the equi-characteristic case, we will also consider deformations of $\mathcal{G}_0$ along

$$k = R'/\mathfrak{m}' \leftarrow R'/\mathfrak{m}'^2 \leftarrow \cdots \leftarrow R'/\mathfrak{m}'^{2^s} \leftarrow R'/\mathfrak{m}'^{2^{s+1}} \leftarrow \cdots \leftarrow R'/\mathfrak{m}'^{2^{\langle \log_2 e \rangle}} \leftarrow R'/\pi R'$$

and deformations of $\mathcal{G}'_1 = \mathcal{G}'_n \otimes_{R'/\pi^n} R'/\pi$ along

$$R'/\pi \leftarrow R'/\pi^2 \leftarrow \cdots \leftarrow R'/\pi^{2^s} \leftarrow R'/\pi^{2^{s+1}} \leftarrow \cdots \leftarrow R'/\pi^{2^{\langle \log_2 n \rangle}} \leftarrow R'/\pi^n.$$

We also use Lemma 3.1.4 to do induction. The only difference from the mixed characteristic case is that N is always p. Similarly, we get an isomorphism $f_n : \mathcal{G}'_n \xrightarrow{\sim} \mathcal{G}_n \otimes_{R/\pi^n} R'/\pi^n$ such that $f_n|_{\mathcal{G}'_n[p^\lambda]} = g|_{\mathcal{G}'_n[p^\lambda]}$.

In both the mixed characteristic and equi-characteristic cases, let $f = f_n$. It suffices to prove that the R'-isomorphism of Barsotti-Tate groups $f : \mathcal{G}'_n \xrightarrow{\sim} \mathcal{G}_n \otimes_R R'$ is Galois equivariant. By Lemma 2.1.7 and Lemma 3.1.2 (2), it suffices to check $f_0 = (f_0)_\sigma$ on k-fibers. Note that $\psi : \sigma \mapsto f_0^{-1} \circ (f_0)_\sigma$ define a cocycle in $C^1(\Gamma, \mathrm{Aut}_{R'}(\mathcal{G}'_0))$. But since R' is totally ramified over R, the Γ-action on $k' = k$ is trivial. Thus, ψ induces a representation $\Gamma \to \mathrm{Aut}_k(\mathcal{G}'_0)$. Now from the condition that $f|_{\mathcal{G}'_n[p^\lambda]} = g|_{\mathcal{G}'_n[p^\lambda]}$ with $g : \mathcal{G}'_n[p^{l_n}] \to \mathcal{G}_n[p^{l_n}] \otimes_R R'$ a Γ-equivariant morphism using the Γ-descent datum for $\mathcal{G}'_n[p^{l_n}]$, we see that $f|_{\mathcal{G}'_0[p^\lambda]}$ is Galois-equivariant. Thus, via ψ the action by Γ on $\mathcal{G}'_0[p^\lambda]$ is trivial. Now we can conclude that ψ is trivial via the following lemma. $\square$

Lemma 3.2.4. Let $\psi \in \mathrm{Aut}_k(\mathcal{G}'_0)$ be an automorphism of finite order. If $\psi|_{\mathcal{G}'_0[p^\lambda]}$ is trivial (recall $\lambda = 1$ for $p > 2$ and $\lambda = 2$ for $p = 2$), then ψ is trivial.

Proof. Passing to the finite order $W(k)$-linear automorphism $\mathbb{D}(\psi)$ of the Dieudonné module $\mathbb{D}(\mathcal{G}_0)$ as a finite free $W(k)$-automorphism, the convergence property of p-adic logarithm on $\mathrm{GL}_{W(k)}(\mathbb{D}(\mathcal{G}_0))$ gives the result. $\square$

4. Proof of the Descent Theorem

4.1. **Messing's theory on deformations of Barsotti-Tate groups.** We now prepare to prove Theorem 1.0.3. Let us first recall some results from [14] on the deformation theory of Barsotti-Tate groups.

Let S be an affine scheme on which p is nilpotent, so there exists an integer $N > 0$ such that p^N kills $\mathcal{O}_S$. Let $\mathcal{G}$ be a Barsotti-Tate group over S.

As an *fppf* abelian sheaf, there exists an extension

$$(4.1.1) \qquad\qquad 0 \to V(\mathcal{G}) \to E(\mathcal{G}) \to \mathcal{G} \to 0$$

of $\mathcal{G}$ by a vector group $V(\mathcal{G})$ (i.e., the group scheme represented by a quasi-coherent locally free sheaf of finite rank). The extension is universal in the sense that for any extension $0 \to M \to \mathcal{E} \to \mathcal{G} \to 0$ of $\mathcal{G}$ by another vector group M there is a unique linear map $V(\mathcal{G}) \xrightarrow{\phi} M$ such that the extension is (uniquely) isomorphic to the pushout of (4.1.1) by ϕ.

In general, we can define a "Lie" functor from the category of group sheaves over S to the category of vector group schemes over S (see §2, Ch. 3, [14]). Taking

"Lie" of the universal extension (4.1.1), Proposition 1.22 in [14] ensures that the following sequence of vector groups over S is exact:

$$0 \to V(\mathcal{G}) \to \mathrm{Lie}(E(\mathcal{G})) \to \mathrm{Lie}(\mathcal{G}) \to 0.$$

For a scheme X, recall that its crystalline site $\mathbf{Crys}(X)$ consists of the category whose objects are pairs $(U \hookrightarrow T, \gamma)$ where:

(1) U is an open sub-scheme of X
(2) $U \hookrightarrow T$ is a locally nilpotent immersion
(3) $\gamma = (\gamma_n)$ are divided powers (locally nilpotent) on the ideal I of U in T

A morphism from $(U \hookrightarrow T, \gamma)$ to $(U' \hookrightarrow T', \delta)$ is a commutative diagram

$$
\begin{array}{ccc}
U & \lhook\joinrel\longrightarrow & T \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\
U' & \lhook\joinrel\longrightarrow & T'
\end{array}
$$

such that $U \to U'$ is the inclusion and $\bar{f} : T \to T'$ is a morphism compatible with divided powers.

Remark 4.1.1. The crystalline site defined above is weaker that what Berthelot defined, but it suffices to study the deformation theory of Barsotti-Tate groups in [14]. We always drop the notation of divided powers γ from the pair $(U \hookrightarrow T, \gamma)$ if no confusion will arise.

Let I be a quasi-coherent ideal of $\mathcal{O}_S$ endowed with locally nilpotent divided powers. Let S_0 be $\mathrm{Spec}(\mathcal{O}_S/I)$, so $(S_0 \hookrightarrow S)$ is an object of the crystalline site of S_0. Let $\mathcal{G}_0, \mathcal{H}_0$ be Barsotti-Tate groups over S_0 and $\mathcal{G}, \mathcal{H}$ respective deformations of $\mathcal{G}_0, \mathcal{H}_0$ over S. (Such deformations always exist when S_0 is affine, by Grothendieck's Theorem (Theorem 3.1.1), because S is necessarily affine when S_0 is affine.) Let $u_0 : \mathcal{G}_0 \to \mathcal{H}_0$ be a morphism of Barsotti-Tate groups. By the construction of universal extension (4.1.1), there exists natural morphism $E(u_0) : E(\mathcal{G}_0) \to E(\mathcal{H}_0)$ induced by u_0. The following results are the technical heart of [14]:

Lemma 4.1.2. *Keep notations as above.*

(1) *There exists a unique morphism $E(u) : E(G) \to E(H)$ which lifts $E(u_0)$.*
(2) *Let $\mathcal{K}$ be a third Barsotti-Tate group on S and $u_0' : \mathcal{H}_0 \to \mathcal{K}_0$ a morphism. Then $E(u_0' \circ u_0) = E(u_0') \circ E(u_0)$.*

Proof. See Chap. IV, §2, [14]. □

Corollary 4.1.3. *Keep notations as above.*

(1) *If $\mathcal{G} = \mathcal{H}$ and $u_0 = \mathrm{Id}_{\mathcal{G}_0}$, then $E(u) = \mathrm{Id}_{\mathcal{G}}$*
(2) *If u_0 is an isomorphism, so is $E(u)$.*

For $\mathcal{G}_0$ a Barsotti-Tate group over S_0, let $\mathbb{D}(\mathcal{G}_0)$ denote $\mathrm{Lie}(E(\mathcal{G}_0))$. Assuming for a moment that there exists a deformation $\mathcal{G}$ of $\mathcal{G}_0$ over S, let $\mathbb{D}(\mathcal{G}_0)_S$ denote $\mathrm{Lie}(E(\mathcal{G}))$. By Lemma 4.1.2, the $\mathcal{O}_S$-module $\mathbb{D}(\mathcal{G}_0)_S$ does not depend on the choice of deformation of $\mathcal{G}_0$ in the sense that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are two deformations of $\mathcal{G}_0$ over S and $E(u) : E(\mathcal{G}_1) \to E(\mathcal{G}_2)$ is the unique isomorphism lifting the identification of $E(\mathcal{G}_0)$ over S_0, then $\mathrm{Lie}(E(u))$ is a natural isomorphism from $\mathrm{Lie}(E(\mathcal{G}_1))$ to $\mathrm{Lie}(E(\mathcal{G}_2))$. In this way, we glue the vector bundles $\mathrm{Lie}(E(\mathcal{G}))$ over open affines in

S (where deformations of $\mathcal{G}_0$ do exist!) to construct a crystal $S \mapsto \mathbb{D}(\mathcal{G}_0)_S$ on S_0 even in the absence of global lifts.

An $\mathcal{O}_S$-submodule $\mathrm{Fil}^1 \hookrightarrow \mathbb{D}(\mathcal{G}_0)_S$ is said to be *admissible* if Fil^1 is a locally-free vector subgroup scheme over S with locally-free quotient such that it reduces to $V(\mathcal{G}_0) \hookrightarrow \mathrm{Lie}(E(\mathcal{G}_0))$ over S_0.

Define a category $\mathscr{C}$ whose objects are pairs $(\mathcal{G}_0, \mathrm{Fil}^1)$ with $\mathcal{G}_0$ a Barsotti-Tate group over S_0 and Fil^1 an admissible filtration on $\mathbb{D}(\mathcal{G}_0)_S$. Morphisms are defined as pairs (u_0, ξ) where $u_0 : \mathcal{G}_0 \to \mathcal{G}_0'$ morphism of Barsotti-Tate groups over S_0 and ξ is morphism of $\mathcal{O}_S$-modules such that the diagram

$$
\begin{CD}
\mathrm{Fil}^1 @>>> \mathbb{D}(\mathcal{G}_0)_S \\
@V{\xi}VV @VV{\mathbb{D}(u_0)_S}V \\
\mathrm{Fil}'^1 @>>> \mathbb{D}(\mathcal{G}_0')_S
\end{CD}
$$

commutes and over S_0 reduces to

$$
\begin{CD}
V(\mathcal{G}_0) @>>> \mathrm{Lie}(E(\mathcal{G}_0)) \\
@V{V(u_0)}VV @VV{\mathrm{Lie}(E(u_0))}V \\
V(\mathcal{G}_0') @>>> \mathrm{Lie}(E(\mathcal{G}_0'))
\end{CD}
$$

Theorem 4.1.4 (Messing). *The functor $\mathcal{G} \mapsto (\mathcal{G}_0, V(\mathcal{G}) \hookrightarrow \mathrm{Lie}(E(\mathcal{G})))$ establishes an equivalence relation of categories:*

$$\text{Barsotti-Tate groups over } S \longrightarrow \mathscr{C}.$$

Proof. Chap. V, §1, [14]. $\qquad\square$

4.2. **Proof of Theorem 1.0.3.** Now we shall use the above theory to prove Theorem 1.0.3. Let us first construct the constant c_0. Recall the constants d and l_n constructed in Proposition 2.3.1 and Theorem 3.2.1 respectively. Set $c_0 = l_d$ in both the mixed characteristic case and the equi-characteristic case. As in Convention 2.2.12, we will state the proof for the mixed characteristic case, and use "(resp.)" to refer the corresponding result in the equi-characteristic case.

Consider the following diagram of deformations:

$$
\begin{CD}
k = R'/\mathfrak{m}' @<<< R'/p @<<< R'/p^2 @<<< \cdots @<<< R'/p^n @<<< R'/p^{n+1} \cdots \\
@| @AAA @AAA @. @AAA @AAA \\
R/\mathfrak{m} @<<< R/p @<<< R/p^2 @<<< \cdots @<<< R/p^n @<<< R/p^{n+1} \cdots
\end{CD}
$$

Respectively, in the equi-characteristic case, we consider the following diagram:

$$
\begin{CD}
k = R'/\mathfrak{m}' @<<< R'/\pi @<<< R'/\pi^2 @<<< \cdots @<<< R'/\pi^n @<<< R'/\pi^{n+1} \cdots \\
@| @AAA @AAA @. @AAA @AAA \\
R/\mathfrak{m} @<<< R/\pi @<<< R/\pi^2 @<<< \cdots @<<< R/\pi^n @<<< R/\pi^{n+1} \cdots
\end{CD}
$$

For each $n \geq 1$, let $\mathcal{G}'_n = \mathcal{G}' \otimes_{R'} R'/p^n$ (resp. $\mathcal{G}'_n = \mathcal{G}' \otimes_{R'} R'/\pi^n$). We have universal extensions on each level n:

$$0 \to V(\mathcal{G}'_n) \to E(\mathcal{G}'_n) \to \mathcal{G}'_n \to 0.$$

Using the "Lie" functor, we get an exact sequence of finite free R'/p^n-modules (resp. R'/π^n-modules)

$$0 \to V(\mathcal{G}'_n) \to \mathrm{Lie}(E(\mathcal{G}'_n)) \to \mathrm{Lie}(\mathcal{G}'_n) \to 0.$$

Put $N'_n = V(\mathcal{G}'_n)$, $M'_n = \mathrm{Lie}(E(\mathcal{G}'_n))$, and $L'_n = \mathrm{Lie}(\mathcal{G}'_n)$. For each n, we have an exact sequence

$$(S'_n) \qquad\qquad 0 \to N'_n \to M'_n \to L'_n \to 0.$$

By the functorial construction of the universal extension (4.1.1), we see that N'_n, M'_n and L'_n are imbued with Γ-descent data coming from the generic fiber of $\mathcal{G}'$ for all n. Furthermore, (S'_n) is a short exact sequence of Γ-descent modules and its formation is compatible with the base changes $R'/p^{n+1} \to R'/p^n$ (resp. $R'/\pi^{n+1} \to R'/\pi^n$).

Lemma 4.2.1. *Under the hypothesis of Theorem 1.0.3, for each $n \geq 1$ the exact sequence of Γ-descent modules (S'_n) has an effective descent*

$$(S_n) \qquad\qquad 0 \to N_n \to M_n \to L_n \to 0$$

such that $(S_{n+1}) \otimes_{R/p^{n+1}} R/p^n \simeq (S_n)$ descending $(S'_{n+1}) \otimes_{R'/p^{n+1}} R'/p^n \simeq (S'_n)$ (resp. $(S_{n+1}) \otimes_{R/\pi^{n+1}} R/\pi^n \simeq (S_n)$ descending $(S'_{n+1}) \otimes_{R'/\pi^{n+1}} R'/\pi^n \simeq (S'_n)$).

Proof. Let $N' = \varprojlim_n N'_n$, $M' = \varprojlim_n M'_n$ and $L' = \varprojlim_n L'_n$. We get a short exact sequence of Γ-descent modules:

$$(S') \qquad\qquad 0 \to N' \to M' \to L' \to 0,$$

where N', M' and L' are finite free R'-modules, and this induces each (S_n) by reduction. It suffices to prove that (S') has an effective descent.

By the hypothesis of Theorem 1.0.3, $\mathcal{G}'[p^{c_0}]$ has an effective descent $\mathcal{H}$. Since $c_0 = l_d$, by Theorem 3.2.1 with $n = d$ there exists a Barsotti-Tate group $\mathcal{G}_d$ over R/p^d (resp. R/π^d) such that $\mathcal{G}_d$ is an effective descent of $\mathcal{G}'_d$. If we take the "Lie" functor of the crystals constructed for $\mathcal{G}'_d$ and $\mathcal{G}_d$, we get an exact sequence of R/p^d-modules (resp. R/π^d-modules)

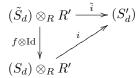$$(\tilde{S}_d) \qquad\qquad 0 \to \tilde{N}_d \to \tilde{M}_d \to \tilde{L}_d \to 0$$

which is an effective descent of

$$(S'_d) \qquad\qquad 0 \to N'_d \to M'_d \to L'_d \to 0.$$

By Proposition 2.3.1, we see that L', M', and N' have effective descents to finite free R-modules L, M and N respectively. By Corollary 2.2.8, (S') has an effective descent to an exact sequence of finite free R-modules

$$(S) \qquad\qquad 0 \to N \to M \to L \to 0.$$

$\square$

Remark 4.2.2. We have to be very careful concerning the following technical point. As two effective descents of (S'_d), the sequences $(\tilde{S}_d)$ and (S_d) are not known to be

isomorphic in the sense that there exists an R-linear isomorphism of *exact sequences* $f : (\tilde{S}_d) \to (S_d)$ such that the following diagram commutes Γ-equivariantly

$$
\begin{array}{ccc}
(\tilde{S}_d) \otimes_R R' & \xrightarrow{\ \tilde{i}\ } & (S'_d) \\
{\scriptstyle f \otimes \mathrm{Id}} \downarrow & \nearrow{\scriptstyle i} & \\
(S_d) \otimes_R R' & &
\end{array}
$$

where $\tilde{i}$ and i are isomorphisms of exact sequences of Γ-descent modules. In fact, Example 2.2.3 shows that $(\tilde{S}_d)$ and (S_d) are not necessarily isomorphic. Fortunately, by Proposition 2.2.4, we see that as effective descents of (S'_{d-r_0}) (resp. (S'_{d-r})),

$$
(\tilde{S}_d) \otimes_R R/p^{d-r_0} \simeq (S_d) \otimes_R R/p^{d-r_0}.
$$

(resp. $(\tilde{S}_d) \otimes_R R/\pi^{d-r} \simeq (S_d) \otimes_R R/\pi^{d-r}$).

Set $m = d - r_0 \geq 1$ (resp. $m = d - r \geq 1$). We shall consider deformations of $\mathcal{G}_m = \mathcal{G}_d \otimes_{R/p^d} R/p^{d-r_0}$ (resp. $\mathcal{G}_m = \mathcal{G}_d \otimes_{R/\pi^d} R/\pi^{d-r}$) along

$$
R/p^m \longleftarrow R/p^{2m} \longleftarrow \cdots \longleftarrow R/p^{sm} \longleftarrow R/p^{(s+1)m} \longleftarrow \cdots
$$

(resp. $R/\pi^m \longleftarrow R/\pi^{2m} \longleftarrow \cdots \longleftarrow R/\pi^{sm} \longleftarrow R/\pi^{(s+1)m} \longleftarrow \cdots$)

Lemma 4.2.3. *For each $s \geq 1$, $\mathcal{G}'_{sm} = \mathcal{G}' \otimes_{R'} R'/p^{sm}$ (resp. $\mathcal{G}'_{sm} = \mathcal{G}' \otimes_{R'} R'/\pi^{sm}$) has an effective descent to an R/p^{sm}- (resp. R/π^{sm}-) Barsotti-Tate group $\mathcal{G}_{sm}$ and there are unique isomorphisms $\mathcal{G}_{(s+1)m} \otimes_R R/p^{sm} \simeq \mathcal{G}_{sm}$ (resp. $\mathcal{G}_{(s+1)m} \otimes_R R/\pi^{sm} \simeq \mathcal{G}_{sm}$) that descend the isomorphisms $\mathcal{G}'_{(s+1)m} \otimes_{R'} R'/p^{sm} \simeq \mathcal{G}'_{sm}$ (resp. $\mathcal{G}'_{(s+1)m} \otimes_{R'} R'/\pi^{sm} \simeq \mathcal{G}'_{sm}$).*

Proof. We only prove the mixed characteristic case. In the equi-characteristic case the proof works if we replace p with π everywhere.

Now let us prove the lemma with by induction on s. For $s = 1$, recall that in the proof of Lemma 4.2.1, by the hypothesis of Theorem 1.0.3 and Theorem 3.2.1 with $n = d$ we get an R/p^d-Barsotti-Tate group $\mathcal{G}_d$ which is an effective descent of $\mathcal{G}'_d$. Letting $\mathcal{G}_m = \mathcal{G}_d \otimes_{R/p^d} R/p^m$ settles the case $s = 1$. For $s = 2$, considering the deformation problem for $\mathcal{G}_m$ along $R/p^m \to R/p^{2m}$, we need to construct a deformation $\mathcal{G}_{2m}$ of $\mathcal{G}_m$ to R/p^{2m} such that $\mathcal{G}_{2m}$ satisfies the conditions of Lemma 4.2.3. We claim that it is equivalent to construct an admissible filtration $\mathrm{Fil}^1 \hookrightarrow \mathbb{D}(\mathcal{G}_m)_{R/p^{2m}}$ such that the following diagram commutes

(4.2.1)
$$
\begin{array}{ccc}
\mathrm{Fil}^1 \otimes_R R' & \hookrightarrow & \mathbb{D}(\mathcal{G}_m)_{R/p^{2m}} \otimes_R R' \\
{\scriptstyle f^1} \downarrow & & \downarrow {\scriptstyle f} \\
V(\mathcal{G}'_{2m}) & \hookrightarrow & \mathrm{Lie}(E(\mathcal{G}'_{2m}))
\end{array}
$$

where f^1, f are Galois-equivariant isomorphisms and the diagram (4.2.1) lifts the following commutative diagram

$$(4.2.2)$$

$$
\begin{array}{ccc}
V(\mathcal{G}_m) \otimes_R R' & \hookrightarrow & \mathrm{Lie}(E(\mathcal{G}_m)) \otimes_R R' \\
\downarrow {\scriptstyle f_0^1} & & \downarrow {\scriptstyle f_0} \\
V(\mathcal{G}'_m) & \hookrightarrow & \mathrm{Lie}(E(\mathcal{G}'_m))
\end{array}
$$

To make notations easier, we denote the above diagrams (e.g., diagram (4.2.1)) in the following way:

$$(\mathrm{Fil}^1 \hookrightarrow \mathbb{D}(\mathcal{G}_m)_{R/p^{2m}}) \otimes_R R' \xrightarrow[\sim]{f} (V(\mathcal{G}'_{2m}) \hookrightarrow \mathrm{Lie}(E(\mathcal{G}'_{2m})))$$

We use Messing's Theorem (Theorem 4.1.4) to prove the above claim concerning the equivalence of our deformation problem and the construction of of (4.2.1). Suppose we can construct such an admissible filtration as in (4.2.1). By Theorem 4.1.4, we have a Barsotti-Tate group $\mathcal{G}_{2m}$ over R/p^{2m} which is a deformation of $\mathcal{G}_m$ such that

$$(\mathrm{Lie}(\mathcal{G}_{2m}) \hookrightarrow \mathrm{Lie}(E(\mathcal{G}_{2m})) \simeq (\mathrm{Fil}^1 \hookrightarrow \mathbb{D}(\mathcal{G}_m)_{R/p^{2m}}).$$

Thus f induces an R'-isomorphism of Barsotti-Tate groups $\tilde{f} : \mathcal{G}_{2m} \otimes_R R' \simeq \mathcal{G}'_{2m}$ which lifts the Galois-equivariant isomorphism $\tilde{f}_0 : \mathcal{G}_m \otimes_R R' \simeq \mathcal{G}'_m$. Now it suffices to check that $\tilde{f}$ is Galois equivariant. By Lemma 2.1.7, we need to show that $\tilde{f}^{-1} \circ \tilde{f}_\sigma = \mathrm{Id}$. Since f, f^1 are Galois equivariant, $f^{-1} \circ f_\sigma$ and $(f^1)^{-1} \circ f^1_\sigma$ are identity maps. By Theorem 4.1.4 and the fact that the diagram (4.2.1) lifts the diagram (4.2.2), it is clear that $\tilde{f}^{-1} \circ \tilde{f}_\sigma = \mathrm{Id}$. This settles one direction of the claim. The proof of the other direction is just reversing the above argument.

Now let us construct an admissible filtration as in the diagram (4.2.1). Recall that for each n, we have an exact sequence of Γ-descent modules

$$(S'_n) \qquad\qquad 0 \to N'_n \to M'_n \to L'_n \to 0$$

where $(N'_n \hookrightarrow M'_n)$ is just $(V(\mathcal{G}'_n) \hookrightarrow \mathrm{Lie}(E(\mathcal{G}'_n)))$. Over R/p^m, by Remark 4.2.2, we do have an isomorphism of effective descents of $(N'_m \hookrightarrow M'_m)$,

$$(N_m \hookrightarrow M_m) \simeq (V(\mathcal{G}_m) \hookrightarrow \mathrm{Lie}(E(\mathcal{G}_m)))$$

Now choose a deformation $\tilde{\mathcal{G}}$ over R/p^{2m} for $\mathcal{G}_m$ (such deformation always exists, due to Theorem 3.1.1). Let $\mathbb{D}(\tilde{\mathcal{G}})_{R/p^{2m}} = \mathrm{Lie}(E(\tilde{\mathcal{G}}))$. From the functorial construction of the crystal $\mathbb{D}(\tilde{\mathcal{G}})_{R/p^{2m}}$, we have that $\mathbb{D}(\tilde{\mathcal{G}})_{R/p^{2m}}$ is a finite free R/p^{2m}-module and $\mathbb{D}(\tilde{\mathcal{G}})_{R/p^{2m}} \otimes_{R/p^{2m}} R/p^m \simeq \mathrm{Lie}(E(\mathcal{G}_m))$. We claim that $\mathbb{D}(\tilde{\mathcal{G}})_{R/p^{2m}}$ is an effective descent for $M'_{2m} = \mathrm{Lie}(E(\mathcal{G}'_{2m}))$. Using Lemma 4.1.2 in our case, let

$$(S_0 \hookrightarrow S) = (\mathrm{Spec}(R'/p^m) \hookrightarrow \mathrm{Spec}(R'/p^{2m})),$$

$$\mathcal{G}_0 = \mathcal{G}_m \otimes_{R/p^m} R'/p^m, \ \mathcal{H}_0 = \mathcal{G}'_m, \ \mathcal{G} = \tilde{\mathcal{G}} \otimes_{R/p^{2m}} R'/p^{2m}, \ \mathcal{H} = \mathcal{G}'_{2m}.$$

We will prove the claim by showing that $E(\tilde{\mathcal{G}})$ is an effective descent of $E(\mathcal{G}'_{2m})$. Since $\mathcal{G}_m$ is an effective descent of $\mathcal{G}'_m$, there exists an Γ-equivariant morphism

$$f_0 : \mathcal{G}_m \otimes_{R/p^m} R'/p^m \xrightarrow{\sim} \mathcal{G}'_m.$$

Corollary 4.1.3 (2) shows that there exists a unique isomorphism

$$E(f_0) : E(\tilde{\mathcal{G}}) \otimes_{R/p^{2m}} R'/p^{2m} \xrightarrow{\sim} E(\mathcal{G}'_{2m})$$

which lifts f_0. Now it suffices to check $E(f_0)$ is Γ-equivariant. By Lemma 2.1.7, we need to check $E(f_0)^{-1} \circ E(f_0)_\sigma = \text{Id}$. But this is clear by Corollary 4.1.3 (1) and the fact that $f_0^{-1} \circ (f_0)_\sigma = \text{Id}$.

As finite free R/p^{2m}-modules, $\mathbb{D}(\tilde{\mathcal{G}})_{R/p^{2m}}$ has the same rank as M_{2m}. Now select an R/p^{2m}-module isomorphism f between $\mathbb{D}(\tilde{\mathcal{G}})_{R/p^{2m}}$ and M_{2m} such that f is the lift of isomorphism $\mathbb{D}(\tilde{\mathcal{G}})_{R/p^{2m}} \otimes_{R/p^{2m}} R/p^m \simeq M_m$. There exists a finite free submodule Fil^1 of $\mathbb{D}(\tilde{\mathcal{G}})_{R/p^{2m}}$ such that the following diagram commutes

(4.2.3)
$$
\begin{array}{ccc}
\mathbb{D}(\tilde{\mathcal{G}})_{R/p^{2m}} & \xrightarrow{\ f\ }_{\sim} & M_{2m} \\
\uparrow & & \uparrow \\
\text{Fil}^1 & \xrightarrow{\ f^1\ }_{\sim} & N_{2m}
\end{array}
$$

where f^1 is the restriction of f on Fil^1.

The map $\text{Fil}^1 \hookrightarrow \mathbb{D}(\tilde{\mathcal{G}})_{R/p^{2m}}$ is an admissible filtration because $L_{2m} \simeq M_{2m}/N_{2m}$ is a finite free R/p^{2m}-module. By Lemma 4.2.1 and the diagram (4.2.3), this admissible filtration is just what we need in the diagram (4.2.1). Therefore, we are done in the case $s = 2$. Note that above construction gives an isomorphism

$$(V(\mathcal{G}_{2m}) \hookrightarrow \text{Lie}(E(\mathcal{G}_{2m}))) \simeq (N_{2m} \hookrightarrow M_{2m})$$

as effective descents of $N'_{2m} \hookrightarrow M'_{2m}$, so we can continue our steps for $s = 3$ and then for any positive integers. $\qquad\square$

Finally, by Lemma 4.2.3, defining $\mathcal{G} = \varinjlim_s \mathcal{G}_s$, we see that $\mathcal{G}$ is an effective descent for $\mathcal{G}'$. Thus, we have proved Theorem 1.0.3.

5. Extensions of Generic Fibres for Finite Flat Group Schemes

5.1. **Proof of the Main Theorem.** In this Chapter, we assume that R is a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field. We will generalize Tate's isogeny theorem [18] and Raynaud' s results (Proposition 2.3.1 in [15]) to finite level. First let us show how Theorem 1.0.5 and Proposition 1.0.6 imply the Main Theorem. Then we will prepare to prove Theorem 1.0.5 and Proposition 1.0.6.

Proof of the Main Theorem. Applying Theorem 1.0.5 and Proposition 1.0.6 to R' and $G' = G \otimes_K K'$, we get constants c_1 and c_2 which depend on $e(K'/\mathbb{Q}_p)$ and the height of G (note that heights of G and G' are the same). Set $c = c_1 + c_2 + c_0 \geq 2$. Suppose that $G[p^c]$ can be extended to a finite flat group scheme $\mathcal{G}_c$. By Proposition 1.0.6, $G[p^{c_1+c_0}]$ can be extended to a truncated Barsotti-Tate group $\mathcal{G}_{c_1+c_0}$. Note that we have $f : \mathcal{G}_{c_1+c_0} \otimes_R K' \simeq \mathcal{G}'[p^{c_1+c_0}] \otimes_{R'} K'$ Galois-equivariantly. By Theorem 1.0.5, $p^{c_1} f$ can be extended an R'-group scheme morphism F'. Note that F' kills the generic fiber of $\mathcal{G}_{c_1+c_0}[p^{c_1}] \otimes_R R'$. Since $\mathcal{G}_{c_1+c_0}[p^{c_1}]$ is flat, F' kills $\mathcal{G}_{c_1+c_0}[p^{c_1}] \otimes_R R'$. Also, it is easy to see that F' factors though $\mathcal{G}'[p^{c_0}]$. Thus, F' induces a morphism $F : \mathcal{G}_{c_1+c_0}[p^{c_0}] \otimes_R R' \to \mathcal{G}'[p^{c_0}]$, and $F \otimes_{R'} K' = f|_{\mathcal{G}_{c_1+c_0}[p^{c_0}]}$ is an isomorphism.

Since f is Galois-equivariant and all group schemes here are R'-flat, any morphism extending f has to be unique. Therefore, F is Galois-equivariant. Using the

same argument for f^{-1}, we see that there exists a morphism

$$F^{-1} : \mathcal{G}'[p^{c_0}] \to \mathcal{G}_{c_1+c_0}[p^{c_0}] \otimes_R R'.$$

Thus, $\mathcal{G}_{c_1+c_0}[p^{c_0}]$ is an effective descent of $\mathcal{G}'[p^{c_0}]$. The Main Theorem then follows by the Descent Theorem (Theorem 1.0.3). $\qquad\square$

5.2. **Dimension of truncated Barsotti-Tate groups.** We will basically follow the similar ideas of Tate and Raynaud to prove our theorem. So we first give an " ad hoc" definition of the dimension for truncated Barsotti-Tate groups and then show that the dimension of truncated Barsotti-Tate groups can read off from the generic fiber if the level is big enough. Just like the case of Barsotti-Tate groups, this fact is crucial to extend the morphisms of generic fibers. Since we will discuss the extension of generic fibers of finite flat group schemes (truncated Barsotti-Tate group) defined over the same base ring, from now on, we fix our base ring R which is a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field and we assume that the absolute ramification index $e = e(K/\mathbb{Q}_p)$ is at least $p - 1$. We made such assumption because by Raynaud's Theorem in [15], if $e < p - 1$, Theorem 1.0.5 and Theorem 1.0.6 automatically works for $c_1 = c_2 = 0$. If G is a scheme over R, let G_K denote the generic fiber of G and same notations applies for the morphism of R-schemes. For a finite flat group scheme G over R, let $\mathrm{disc}(G)$ denote the discriminant ideal of G and $|G|$ denote the order of G. Finally, let $f : G \to H$ be a morphism of R-group schemes, we call f is a *generic isomorphism* if $f_K = f \otimes_R K$ is an K-isomorphism.

Now let us introduce an invariant of truncated Barsotti-Tate groups, the dimension. Given a truncated Barsotti-Tate groups G over R, let $G_k = G \otimes_R k$ be its closed fiber, $G_k^{(p)} = G_k \otimes_{k,\phi} k$, where ϕ is Frobenius. There exists a canonical k-group scheme morphism (relative Frobenious) $F : G_k \to G_k^{(p)}$ which kernel is finite flat k-group scheme and $\mathrm{Ker}(F) \hookrightarrow G[p]$ is a closed immersion.

Definition 5.2.1. The dimension of G is $\log_p |\mathrm{Ker}(F)|$. We write $\dim(G)$ or $\mathrm{d}(G)$ to denote it.

Let us "justify" Definition 5.2.1 by checking compatibility with the definition of dimension for Barsotti-Tate groups. By Grothendieck's Theorem in [17], for any truncated Barsotti-Tate group G over R of level n, there exists a Barsotti-Tate group $\mathcal{G}$ such that $G = \mathcal{G}[p^n]$. We also have $\dim(\mathcal{G}) = \log_p |\mathrm{Ker}(F)|$ where $F : \mathcal{G}_k \to \mathcal{G}_k^{(p)}$ is the relative Frobenius on $\mathcal{G}_k$, so $\dim(G) = \dim(\mathcal{G})$.

The following proposition show that if level n is big enough, the generic fiber of truncated Barsotti-Tate group will decide the dimension. For any x is a real number, recall that $[x] = \mathrm{Max}\{m|m$ is an interger such that $m \leq x\}$ and $\lceil x \rceil = \mathrm{Min}\{m|m$ is an integer such that $m \geq x\}$.

Proposition 5.2.2. *Let G and H be two truncated Barsotti-Tate group of level n over R which have the same generic fiber. There exists $r_2 \geq 2$ depending only on e and height h such that if $n \geq r_2$ then $\mathrm{d}(G) = \mathrm{d}(H)$.*

Proof. It suffices to prove the proposition on the strictly henselization of R. So we can assume that k is algebraically closed. From Grothendieck's Theorem in [17], there exist Barsotti-Tate groups $\mathcal{G}$ and $\mathcal{H}$ such that $G = \mathcal{G}[p^n]$ and $H = \mathcal{H}[p^n]$. In [15], Raynaud proved that as one dimensional Galois module $\det(T_p(\mathcal{G})) \simeq \epsilon^d$,

where ϵ is the cyclotomic character, $T_p(\mathcal{G})$ is the Tate module of $\mathcal{G}$ and $d = \dim(\mathcal{G})$. Since generic fibers of G and H are the same, we have

$$\epsilon^{\mathrm{d}(G)} = \det((T_p(\mathcal{G})) \equiv \det(T_p(\mathcal{H})) = \epsilon^{\mathrm{d}(H)} \mod (p^n)$$

Let λ be the maximal integer such that $\mathbb{Q}_p(\zeta_{p^\lambda}) \subseteq K$, where ζ_{p^λ} is p^λ-th root of unity. Then there exists a $\sigma \in \mathrm{Gal}(\overline{K}/K)$ such that $\epsilon(\sigma) = 1 + p^\lambda u$, where u is a unit in $\mathbb{Z}_p$. Consider the action of σ on the congruence relation above, we get

$$(1 + p^\lambda u)^{\mathrm{d}(G)} \equiv (1 + p^\lambda u)^{\mathrm{d}(H)} \mod (p^n).$$

That is, $\mathrm{d}(G) \equiv \mathrm{d}(H) \mod (p^{(n-\lambda)})$. Note that $e(\mathbb{Q}_p(\zeta_{p^\lambda}) : \mathbb{Q}_p) \leq e$, we have $\lambda \leq [\log_p(\frac{e}{p-1})] + 1$. Set $r_2 = \lceil \log_p h \rceil + [\log_p(\frac{e}{p-1})] + 1$. we have

$$\mathrm{d}(G) \equiv \mathrm{d}(H) \mod (p^{(n-\lambda)}) \equiv (p^{\lceil \log_p h \rceil})$$

By proposition 6.2.8 in [6], we know that $\mathrm{d}(G), \mathrm{d}(H) \leq h$, so $\mathrm{d}(G) = \mathrm{d}(H)$. $\square$

From above Proposition and using the same argument of Tate (Proposition 6.2.12, [6] or [18]), for truncated Barsotti-Tate groups G of level n, $\mathrm{disc}(G) = p^{dnp^{nh}}$ for $n \geq r_2$, where d is the dimension of G. Thus we have following corollary:

Corollary 5.2.3. *Let $f : G \to H$ be a morphism of truncated Barsotti-Tate groups of level with f_K an isomorphism. If $n \geq r_2$ then f is an isomorphism.*

The following lemma is a useful fact on the scheme-theoretic closure of group schemes.

Lemma 5.2.4. *Let $f : G \to H$ be a morphism of finite flat group schemes over R, $G_K^{(1)}$, $H_K^{(1)}$ be K-subgroup schemes of G_K, H_K respectively. Let $G^{(1)}$ and $H^{(1)}$ be the scheme-theoretic closures of $G_K^{(1)}$, $H_K^{(1)}$ in G and H respectively. Suppose that $f \otimes_R K|_{G_K^{(1)}}$ factor through $H_K^{(1)}$. Then $f|_{G^{(1)}}$ factors through $H^{(1)}$.*

Proof. Let $G' = G \times_{f,H,i} H^{(1)}$, where $i : H^{(1)} \hookrightarrow H$ be the closed immersion. We see that G' is a closed subgroup scheme of G. It suffices to show that the generic fiber of G' contains $G_K^{(1)}$. Since $f_K|_{G_K^{(1)}}$ factor through $H_K^{(1)}$, this is clear. $\square$

Corollary 5.2.5. *Notations as above, let G_i, H_i be scheme-theoretic closures of $G_K[p^i]$, $H_K[p^i]$ in G , H respectively, then $f|_{G_i}$ factors through H_i.*

5.3. **Proof of Proposition 1.0.6 and Theorem 1.0.5.** Now we are ready to prove Proposition 1.0.6 and Theorem 1.0.5 First we need to bound discriminants of finite flat group schemes of type $(p, \cdots, p)$ in terms of heights and the absolute ramification index $e(K/\mathbb{Q}_p)$.

Let G be a finite flat group scheme over R killed by p. If $|G| = p^h$, we define h be the *height* of G. Note that if $\mathcal{G}$ is a Barsotti-Tate group (or a truncated Barsotti-Tate group) with height h. Then $\mathcal{G}[p]$ will has height h as a finite flat group scheme killed by p. Recall that $\mathrm{disc}(G)$ denote the discriminant ideal of G.

Lemma 5.3.1. *With notations as above, $\mathrm{disc}(G) = (a)^{p^h}$, where $a \in R$ and $a|p^h$.*

Proof. Let $0 \to G_1 \to G_2 \to G_3 \to 0$ be an exact sequence of finite flat group schemes. By Lemma 6.2.9 [6], we have

$$(5.3.1) \qquad \operatorname{disc}(G_2) = \operatorname{disc}(G_1)^{|G_3|} \operatorname{disc}(G_3)^{|G_1|}$$

Now passing to the connected-étale sequence $0 \to G^0 \to G \to G^{\text{ét}} \to 0$, since $\operatorname{disc}(G^{\text{ét}}) = 1$, we see that $\operatorname{disc}(G) = \operatorname{disc}(G^0)^{|G^{\text{ét}}|}$. Thus, we reduce the problem to the case that G is connected.

Now since G is connected, by Theorem 3.1.1 in [2] there exists an isogney f between formal groups $\mathcal{G}, \mathcal{G}'$ such that

$$0 \longrightarrow G \longrightarrow \mathcal{G} \stackrel{f}{\longrightarrow} \mathcal{G}' \longrightarrow 0$$

is exact. By Corollary 4.3.10 in [6], $\operatorname{disc}(G) = \mathrm{N}_{\mathcal{O}_{\mathcal{G}}/f^*\mathcal{O}_{\mathcal{G}'}}(a)$, where $\mathcal{O}_{\mathcal{G}}$ and $\mathcal{O}_{\mathcal{G}'}$ are formal Hopf algebras of $\mathcal{G}$ and $\mathcal{G}'$ respectively; (a) is the principal ideal in $\mathcal{O}_{\mathcal{G}'}$ for the annihilator of cokernel $f^*(\Omega^n_{\mathcal{G}'}) \to \Omega^n_{\mathcal{G}}$ and $\mathrm{N}_{\mathcal{O}_{\mathcal{G}}/f^*\mathcal{O}_{\mathcal{G}'}}(\cdot)$ is the norm.

Let d be the common dimension of $\mathcal{G}$ and $\mathcal{G}'$. By Theorem 3.2.1 in [6], we can choose basis of invariant differentials $w'_1, \cdots, w'_d$ and $w_1, \cdots, w_d$ on $\mathcal{G}'$ and $\mathcal{G}$ respectively. Since f is an isogeny, for each $i = 1, \ldots, d$, $f^*(w'_i)$ is an invariant differential on $\mathcal{G}$, thus $f^*(w'_i) = \sum_{j=1}^d a_{ij} w_j$ for $a_{ij} \in R$. Define $a = \det(a_{ij}) \in R$, so $\operatorname{disc}(G) = \mathrm{N}_{\mathcal{O}_{\mathcal{G}}/f^*\mathcal{O}_{\mathcal{G}'}}(a) = (a)^{p^h}$.

Now let us show that $a | p^h$. First let us reduce the problem to the case that the generic fiber of G is simple. In fact, suppose G'_K is a nontrivial subgroup of G_K, let G' be the scheme-theoretic closure of G'_K in G. We get an exact sequence of finite flat group schemes

$$0 \to G' \to G \to G'' \to 0.$$

Then by formula (5.3.1), we reduce the problem to the case which the generic fiber G is simple.

In general, if $G' \to G$ is a morphism of R-finite flat group schemes that is an isomorphism on K-fibers, we have $\operatorname{disc}(G) | \operatorname{disc}(G')$. Thus, if we fix the generic fiber G_K of G it suffices to prove $a | p^h$ for the *minimal model* $G_{\min}$ of G_K in the sense of Raynaud. If G_K is simple, then by [15] such a minimal model is an **F**-group scheme of rank 1 for some finite field **F**. Since there is a connected model of G, clearly $G_{\min}$ must be connected. Thus it suffices to prove that $a | p^h$ where G is a connected **F**-group scheme of rank 1.

If we normalize the valuation by putting $v(p) = e$, according to Theorem 1.4.1 in [15] the Hopf algebra $\mathcal{O}_G$ of a connected **F**-group scheme has the following shape:

$$R[\![X_1, \cdots, X_h]\!]/(X_i^p - \delta_i X_{i+1}, i = 1, \ldots, h)$$

where $\delta_i \in R$ satisfies $1 \le v(\delta_i) \le e$ for each $i = 1, \ldots, h$.

Using Theorem 4.3.9 in [6], the *different* of $\mathcal{O}_G$ is the ideal of $R[\![X_1, \cdots, X_h]\!]$ generated by $\det(\frac{\partial(X_i^p - \delta_i X_{i+1})}{\partial X_i})$. However,

$$\det\left(\frac{\partial(X_i^p - \delta_i X_{i+1})}{\partial X_i}\right) = \begin{vmatrix} pX_1^{p-1} & -\delta_1 & \cdots & 0 \\ 0 & pX_2^{p-1} & \ldots & 0 \\ \cdots & & & \\ -\delta_h & \cdots & & pX_h^{p-1} \end{vmatrix} = p^h \prod_{i=1}^h X_i^{p-1} - \prod_{i=1}^h \delta_i,$$

so

$$\mathrm{disc}(G) = \mathrm{N}_{\mathcal{O}_G/R}(p^h \prod_{i=1}^{h} X_i^{p-1} - \prod_{i=1}^{h} \delta_i).$$

Note that for any X_i, we have $X_i \cdot (X_1^{p-1} \cdots X_h^{p-1}) = \delta_1 \cdots \delta_h X_i$ in $\mathcal{O}_G$. Thus if we select the R-basis of $\mathcal{O}_G$ given by $\{\prod_{i=1}^{h} X_i^{n_i}, 0 \le n_i \le p-1\}$ then for any $\prod_{i=1}^{h} X_i^{n_i}$ with some $n_i > 0$ we have the following equality in $\mathcal{O}_G$:

$$(\prod_{i=1}^{h} X_i^{n_i}) \cdot (p^h \prod_{i=1}^{h} X_i^{p-1} - \prod_{i=1}^{h} \delta_i) = (p^h - 1) \prod_{i=1}^{h} \delta_i (\prod_{i=1}^{h} X_i^{n_i}).$$

Thus, $\mathrm{disc}(G) = (-(p^h - 1)^{p^h - 1}(\delta_1 \cdots \delta_h)^{p^h})$. Finally, since $v(\delta_i) \le e$ for each i we have $\mathrm{disc}(G) = (\delta_1 \cdots \delta_h)^{p^h}$ and $\delta_1 \cdots \delta_h | p^h$. $\qquad \square$

Proof of Proposition 1.0.6. For each $1 \le i \le n$, let G_i be the scheme theoretic closure of $G_K[p^i]$ in G, so G_i is a closed subgroup scheme of G_{i+1}.

By Corollary 5.2.5, we see that p^m induces a morphism $G_{i+m} \to G_i$. Furthermore, p^m induces morphisms $\psi_{m,i,l}: G_{i+m+l}/G_{i+m} \to G_{i+l}/G_i$ which are generic isomorphisms. Therefore, if D_i is the affine algebra of G_{i+1}/G_i, then for each i, $\psi_{1,1,i}: D_i \to D_{i+1}$ induces an isomorphism $D_i \otimes K \simeq D_{i+1} \otimes K$. If A denotes the common affine algebra of the $(D_i \otimes K)$'s, then we see by flatness that $D_i \to A$ is injective, so we can identify D_i with its image in A. This gives a compatible system of injections $D_i \hookrightarrow D_{i+1}$. Therefore, we see that D_i is an increasing sequence of finitely generated R-submodules of the finite étale K-algebra A. Since $D_i \subset D_{i+1}$ inside A, we have $\mathrm{disc}(D_{i+1})|\mathrm{disc}(D_i)\ldots|\mathrm{disc}(D_1)$. By Lemma 5.2.2, we see that $\mathrm{disc}(D_i) = (a_i)^{p^h}$ and $a_{i+1}|a_i|p^h$ for each $1 \le i \le n$. Thus, the number of total possible distinct D_i is at most eh.

Set $r_1 = r_2(eh) + 1$. If $n \ge r_1$, then there exists i_0 such that

(5.3.2) $$D_{i_0} = D_{i_0+1} = \cdots = D_{i_0+s} \text{ and } s \ge r_2.$$

Let $\Gamma_m = G_{i_0+m}/G_{i_0}, m = 1, \ldots, s$. We claim that (Γ_m) is a truncated Barsotti-Tate group of level s.

In fact, the closed immersions $G_i \hookrightarrow G_{i+1}$ induce closed immersions $i_m: \Gamma_m \hookrightarrow \Gamma_{m+1}$. Since $(\Gamma_m)_K \simeq (G_K)[p^m]$, (Γ_m) is a generic truncated Barsotti-Tate group of level s. Thus, all we have to check is that i_m identifies Γ_m with $\Gamma_{m+1}[p^m]$ for all $m = 0, \ldots, s-1$. Consider the diagram

(5.3.3)
$$\begin{array}{ccccc}
\Gamma_{m+1} = G_{i_0+m+1}/G_{i_0} & \xrightarrow{p^m} & G_{i_0+m+1}/G_{i_0} = \Gamma_{m+1} \\
\downarrow{\alpha} & & \uparrow{\gamma} \\
G_{i_0+m+1}/G_{i_0+m} & \xrightarrow{\beta} & G_{i_0+1}/G_{i_0},
\end{array}$$

where α is the canonical projection, β is the map induced by p^m, and γ is the canonical closed immersion. Checking on the generic fiber, we see that (5.3.3) commutes over R. On the other hand, by our choice of i_0 we see that β is an isomorphism, and therefore $\ker[p^m] = \ker \alpha$, which is nothing more than Γ_m. This proves the claim that (Γ_m) is a truncated Barsotti-Tate group of level s.

Let I be the set of indices $\{i | D_i = D_{i+1} = \cdots = D_{i+s_i}, \ s_i \ge r_2\}$, i'' and i' the maximal and minimal elements of I respectively. Let $\Gamma''_m = G_{i''+m}/G_{i''}$, for

$m = 0, \ldots, r_2$ and $\Gamma'_m = G_{i'+m}/G_{i'}$, for $m = 0, \ldots, r_2$. We see that (Γ''_m) and (Γ'_m) are truncated Barsotti-Tate groups over R of level r_2. Since $p^{i''-i'}$ induces a generic isomorphism $\Gamma''_m \to \Gamma'_m$, from Proposition 5.2.2 we have $\mathrm{d}(\Gamma'') = \mathrm{d}(\Gamma') = d$, so $D_{i''} = D_{i'} = p^{dp^h}$. Thus, we have

$$(5.3.4) \qquad\qquad D_{i'} = D_{i'+1} = \cdots = D_{i''} = \cdots = D_{i''+s_{i''}}.$$

We claim that there are at least $n - r_1 + r_2 + 1$ terms D_i in (5.3.4). In fact, the size of set of indices $I' = \{i \mid i < i' \text{ or } i > i'' + r_2\}$ cannot be larger than $(eh-1)r_2$ because otherwise, since we have at most eh many possible distinct D_i, there would exist a chain $D_i = D_{i+1} = \cdots = D_{i+r_2}$ with $\{i, \ldots, i+r_2\} \subset I'$, contradicting the definition of i' and i''. Thus, there are at least $n - (eh-1)r_2 = n - r_1 + r_2 + 1$ terms D_i in (5.3.4). Now set $n_1 = n - r_1 + r_2$. Let $G'_m = G_{i'+m}/G_{i'}$, for $m = 0, \ldots, n_1$. Repeating the proof above, we see that (G'_m) is a truncated Barsotti-Tate group of level n_1 whose generic fiber is $G_K[p^{n_1}]$.

Set $G' = G_{i'+n_1}/G_{i'}$. We see that $p^{i'}$ induces a generic isomorphism $\tilde{g} : G' \to G_{n_1}$. Let g' be the composite of $\tilde{g}$ and the closed immersion $G_{n_1} \hookrightarrow G$. We see that g'_K factors though $G_K[p^{n_1}]$, and $g'_K : G' \to G_K[p^{n_1}]$ is an isomorphism.

By Corollary 5.2.5, $p^{r_1-r_2-i'} : G \to G$ factors through $G_{i'+n_1}$, so composing with the natural projection $G_{i'+n_1} \to G_{i'+n_1}/G_{i'} = G'$ gives a morphism $g : G \to G'$ such that $g \otimes_R K = p^{r_1-r_2}$.

Finally, set $c_2 = r_1 - r_2$. The above proof settles the case $n \geq r_1$. For the case $r_1 \geq n \geq c_2$, let us return to (5.3.2). Since $r_2 \geq n - c_2$, it is easy to see that there exists i_0 such that

$$D_{i_0} = D_{i_0+1} = \cdots = D_{i_0+s} \text{ and } s \geq n - c_2.$$

Let $\Gamma_m = G_{i_0+m}/G_{i_0}$ for $m = 1, \ldots n - c_2$. We see that (Γ_m) is a truncated Barsotti-Tate group of level $n - c_2$. The rest of proof is the same as of the case $n \geq r_1$. $\square$

Proof of Theorem 1.0.6. Set $c_1 = r_1$. Let M be the graph of f in $G_K \times H_K$, i.e., the scheme-theoretic image of $1 \times f : G_K \to G_K \times H_K$. It is obvious that M is a truncated Barsotti-Tate group over K of level n. Using the natural projections of $G_K \times H_K$ to G_K and H_K, we have natural K-group scheme morphisms $p_1 : M \to G_K$ and $p_2 : M \to H_K$. Obviously, p_1 is an isomorphism and $p_2 \circ p_1^{-1} = f$.

For each $1 \leq i \leq n$, let E_i be the scheme-theoretic closure of $M[p^i]$ in $G \times_R H$. From §2, [15], we see that $E_i \hookrightarrow E_{i+1}$ is a closed immersion and E_i is a closed subgroup scheme of $G \times_R H$. We therefore have morphisms of R-group schemes $\hat{p}_1 : E_n \to G$ and $\hat{p}_2 : E_n \to H$ which extend morphisms $p_1 : M \to G_K$ and $p_2 : M \to H_K$ of generic fibers. Because $n \geq r_1$, from Proposition 1.0.6 we see that there exists a truncated Barsotti-Tate group E' of level $n_1 = n - r_1 + r_2$ and a morphism $g' : E' \to E_n$ such that $g'_K : E' \to (E_n)_K[p^{n_1}]$ is an isomorphism, so $\hat{p}_1 \circ g' : E' \to G[p^{n_1}]$ is an isomorphism on generic fibers. By Corollary 5.2.3, since $n_1 = n - r_1 + r_2 \geq r_2$ we see that $\hat{p}_1 \circ g'$ is an isomorphism. Finally, set $F = \hat{p}_2 \circ g' \circ (\hat{p}_1 \circ g')^{-1} \circ p^{c_2}$. We are done. $\square$

6. Application to Abelian Varieties

6.1. **Proof of Theorem 1.0.2.** In this chapter, we assume that R is a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field, and

K is the fraction field. Let A be an abelian variety defined over K with dimension g. Suppose A has potentially good reduction. For example, A can be taken to be an abelian variety with complex multiplication (§3, [16]). Theorem 1.0.2 follows from the Main Theorem and the following lemma.

Lemma 6.1.1. *Suppose A is an abelian variety over K of dimension g with potentially good reduction. There exists a constant c_3 which depends only on g such that there exists a finite Galois extension K' of K with $[K' : K] \leq c_3$ and A acquires good reduction over K'.*

Proof. Choose a prime $l \in \{3, 5\}$ with $l \neq p$. Let $\rho_l : \mathrm{Gal}(K^{\mathrm{s}}/K) \to \mathrm{Aut}(T_l(A))$ be the Galois representation. From §1, [16], we know that A has potentially good reduction over K if and only if $\mathrm{Im}(\rho_l)$ in $\mathrm{Aut}(T_l(A))$ is finite. Since $\mathrm{Id} + l\mathrm{Mat}_{2g}(\mathbb{Z}_l)$ has no torsion as multiplicative group, we see that $\mathrm{Id} + l\mathrm{Mat}_{2g}(\mathbb{Z}_l)$ meets the finite image of ρ_l trivially. Thus $\bar{\rho}_l : \mathrm{Gal}(K^{\mathrm{s}}/K) \to \mathrm{Mat}_{2g}(\mathbb{Z}/l\mathbb{Z})$ is injective. Therefore, the size of $\mathrm{Im}(\rho_l)$ is bounded by the size of $\mathrm{Mat}_{2g}(\mathbb{Z}/l\mathbb{Z})$ which is l^{4g^2}. Let K' be the fixed field by $\mathrm{Ker}(\rho_l)$, so A acquires good reduction over K' with $[K' : K] = l^{4g^2}$. Finally, we can put $c_3 = \mathrm{Max}\{3^{4g^2}, 5^{4g^2}\}$. $\qquad\square$

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