Group schemes I

Def S = a base scheme (preferably affine, or even a field)

A *group scheme* $/S$ is a group object in the cat. of schemes $/S$.

ie an S-scheme G + maps
- $m : G \times_S G \to G$ [multiplication]
- $i : G \to G$ [inversion]
- $e : S \to G$ identity section,

such that the following diagrams commute:

(1) [assoc.]
$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "m \times 1_G"] \arrow[d, "1_G \times m"'] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"'] & G \end{tikzcd}$$

2) [neutral el.]
$$\begin{tikzcd} S \times_S G \arrow[d, "e \times 1_G"'] & G \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, equal] & G \times_S S \arrow[d, "1_G \times e"] \\ G \times_S G \arrow[r, "m"'] & G & G \times_S G \arrow[l, "m"] \end{tikzcd}$$

(3) [inverses]
$$\begin{tikzcd} G \arrow[rr, "{(id, i)}"] \arrow[dr, "str"] \arrow[dd, "{(i, id)}"'] & & G \times_S G \arrow[dd, "m"] \\ & S \arrow[dr, "e"] & \\ G \times_S G \arrow[rr, "m"'] & & G \end{tikzcd}$$

(4) G is commutative if further
$$\begin{tikzcd} G \times_S G \arrow[rr, "swap"] \arrow[dr, "m"'] & & G \times_S G \arrow[dl, "m"] \\ & G & \end{tikzcd}$$

Functor of points description

1) Given a group scheme G, the rep. functor $Hom_S(-, G) : Sch_S \to Sets$ naturally lifts to a functor $Hom_S(-, G) : Sch_S \to Grp$

ie $\forall\, Y \in Sch/S$, $G(Y) := Hom_S(Y, G)$ is a group and $G(Y) \to G(Y')$ is a group hom $\forall\, Y' \to Y$

2) Conversely: A lift of $Hom_S(-, G) : Sch/S \to Sets$ to a functor $Sch/S \Longrightarrow Grp$ defines a structure of a group scheme on G, by Yoneda lemma.

[Ex: To recover $m: G\times G \to G$, one can consider $\mu: \underbrace{h_G \times h_G}_{= h_{G\times G}} \Rightarrow h_G$.

Then $\mu_{G\times G}: h_{G\times G}(G\times G) \to h_G(G\times G)$ takes $1_{G\times G}$ to m.
$$\begin{array}{ccc} h_{G\times G}(G\times G) & \to & h_G(G\times G) \\ \| & & \| \\ \mathrm{Hom}(G\times G, G\times G) & \to & \mathrm{Hom}(G\times G, G) \end{array}$$
]

Finite flat group schemes & Hopf algebras.

Assume S (= Spec(R)) is affine & Noetherian. Let G be a finite & flat group scheme over $S \Rightarrow G$ is itself affine, noetherian, $G = \mathrm{Spec}\, A$ with A/R finite and flat $\Rightarrow$ finite & projective.

The order of G is the function $x \in \mathrm{Spec}(R) \mapsto \mathrm{rank}_{R_{p_x}} A_{p_x}$ ... locally constant

When $R = k$ is a field, this is simply ~~dim~~ the number $\# G = \dim_k A$.

Since $G = \mathrm{Spec}\, A$ is affine, the group operations become

(1) comultiplication $A \xrightarrow{\mu} A \otimes_R A$,

(2) antipode map $A \xrightarrow{i} A$, + we have dual comm. diagrams

(3) counit $A \to R$,

$\rightsquigarrow$ obtain projective, R-Hopf-algebra.
& commutative

Examples of grp schemes

- typical non-affine grp schemes are Abelian varieties / k
- affine - algebraic groups (ie ~~matrix~~ classical matrix groups etc.)
  - additive group $\mathbb{G}_a = \mathrm{Spec}\, R[T]$

    Functor of pts: $\mathbb{G}_a(B) = (B, +)$

    Hopf algebra structure: $R[T] \to R[T] \otimes_R R[T] \cong R[X, Y]$

    $T \longmapsto X+Y$

- multiplicative group $\mathbb{G}_m = \operatorname{Spec} R[T^{\pm 1}]$

$$\mathbb{G}_m(B) = (B^\times, \cdot)$$

$$R[T^{\pm 1}] \longrightarrow R[x^{\pm 1}, y^{\pm 1}]$$

$$T \longmapsto xy$$

- Constant groups G abstract group (say finite)

$$\rightsquigarrow \underline{G} = \coprod_{g \in G} \operatorname{Spec} R$$

$$\underline{G}(X) = \operatorname{Hom}(X, \underline{G}) \simeq \operatorname{Set}(\underbrace{\pi_0(X)}_{\text{set of components}}, G) \simeq \underbrace{G^{\times \pi_0(X)}}_{\text{has natural group structure}}$$

Hopf comultiplication: $\prod_g R\, e_g \longrightarrow \prod_{g_1, g_2} R e_{g_1} \otimes R e_{g_2}$

$$e_g \longmapsto \sum_{g_1 g_2 = g} e_{g_1} \otimes e_{g_2}$$

- this one is finite flat (when G is finite)

Kernels, cokernels, quotients

- Kernels exist easily in the category of group schemes:

$$\begin{tikzcd} G_1 \arrow[r, "f"] & G_2 \\ \ker f \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & S \arrow[u, "e"'] \end{tikzcd}$$

  ! even when G_1, G_2 are finite flat, $\ker f$ is finite, but may fail to be flat.
  (over a field k no problem.)

- Cokernels do not exist in general but do for $f: G_1 \to G_2$ when G_i are commutative finite flat and $\ker f$ is flat.
  - in particular, over a field k the cat. of finite flat comm. group schemes has kernels & cokernels, and in fact is an abelian category.
  - In terms of pts $\operatorname{coker} f$ is the sheafification of $T \mapsto G_2(T)/G_1(T)$ wrt fppf

More examples

(1) A a.v. $/k$ $\rightsquigarrow$ $A[n] := \ker\left(A \xrightarrow{[n\cdot -]} A\right)$ is a finite flat commutative group scheme.

(order $= n^{2g}$ if char $k \nmid n$)

(2) $\mu_n = \ker\left(\mathbb{G}_m \xrightarrow{(\,)^n} \mathbb{G}_m\right)$. $\mu_n \simeq \operatorname{Spec} R[T]/(T^n - 1)$

(3) Assume that char $k = p > 0$. Then

$\mathbb{G}_{a/k}$ has a relative Frobenius on it, $Fr: \mathbb{G}_a \to \mathbb{G}_a$ given by

$$\begin{aligned} k[T] &\to k[T] \\ T &\mapsto T^p \end{aligned}$$

This is a group hom., on B-points it induces $B^* \to B$, which is additive
$$x \mapsto x^p$$

$\rightsquigarrow$ Let $\alpha_p = \ker(Fr: \mathbb{G}_a \to \mathbb{G}_a)$, so $\alpha_p = \operatorname{Spec} k[T]/(T^p)$

<u>Note:</u> $\alpha_p \simeq \mu_p$ as schemes $\left(k[T]/(T^p - 1) = \dfrac{k[T]}{(T-1)^p} \simeq \dfrac{k[T]}{(T^p)} \right)$,
but not as group schemes

Étale finite group schemes

<u>Recall</u> ("Galois theory for schemes")

Given a connected base scheme S and a geometric pt $s: \operatorname{Spec} \Omega \to S$,
the geometric fiber functor yields an equivalence $\underbrace{FEt_S}_{\text{finite étale schemes}/S} \xrightarrow{\sim} \pi_1(S, s)\text{-sets}$

$$Y \to S \longmapsto Y_s$$

(more precisely, $Y_s \to \operatorname{Spec} \Omega$ is finite & étale $\Rightarrow$ $Y_s = \Omega_1 \times \dots \times \Omega_k$ since Ω is separably closed. The étale fund. group is abstractly the group of automorphisms of the fiber functor, hence it tautologically acts on Y_s)

When we restrict from FEt'_S to finite ~~flat~~ étale commutative group schemes/S

(= commutative group objects in FEt'_S), the corresponding RHS also consists of commutative group objects = $\pi_1(S,s)$-modules (= $\pi_1(S,s)$-sets with compatible group structure).

When $S = \operatorname{Spec} k$ for a field k, $\pi_1(S,\bar{s})$ is naturally identified with $\operatorname{Gal}(k^s/k) =: G_k$ so ~~étale~~ finite étale commutative group schemes / $\operatorname{Spec} k \xrightarrow{\sim}$ finite discrete continuous G_k-modules.

Connected component

Let $S = \operatorname{Spec}(R)$ with R local henselian

<u>Recall</u>: this means ~~that~~ equivalently that every finite R-algebra is a finite product of local rings.

$\rightarrow$ a finite flat grp scheme/S is of the form $\operatorname{Spec} A_0 \sqcup \operatorname{Spec} A_1 \sqcup \dots \sqcup \operatorname{Spec} A_n$ where A_i are finite, flat & local R-algebras.

$\operatorname{Spec} R$ is connected $\Rightarrow$ the identity section lands in one component, say $\operatorname{Spec}(A_0)$. One can show that $G^0 = \operatorname{Spec}(A_0)$ is closed under m, i, and e $\rightarrow$ it's a ~~cl~~ open subgroup scheme, called the <u>connected component of G</u>.

The quotient G/G^0 ~~(~~ (= $\operatorname{Coker} G^0 \hookrightarrow G$) exists, and this quotient contains the trivial group $\underbrace{G^0/G^0}_{\cong \operatorname{Spec} R}$ as a connected component

By translation argument, any other con. component is isomorphic to this one

$\Rightarrow G/G^0 = \coprod_{i=1}^{n} \operatorname{Spec} R \rightarrow \operatorname{Spec} R$ is étale.

$\rightsquigarrow$) we obtain the connected-étale sequence

$$0 \to G^0 \hookrightarrow G \to G/G^0 = G^{ét} \to 0.$$

Facts: 1) when ~~G/k~~ k is perfect field as a base, the sequence uniquely splits.

2) when $\# G$ is invertible, then G is étale.

$\# G$ invertible in $R \Rightarrow G$ is étale:

1) WLOG $R = k$, is a field: G is flat, so deciding whether it is étale or not is done fiberwise

$\rightsquigarrow$ may consider point(s) $\operatorname{Spec} k \hookrightarrow R$ and G_k. Then order (becomes constant and) is invertible in k.

2) ~~Consider the~~ de Let G be connected, so $G = \operatorname{Spec} A$ with A local.

write $A = k \oplus I$, consider the derivation

$$\pi : A = k \oplus I \to I \twoheadrightarrow I/I^2.$$

Lift a basis of I/I^2 to $x_1 \dots x_n \in I$. Define derivations $D_i : A \to A$

as $A \xrightarrow{\mu} A \otimes A \xrightarrow{id \otimes \pi} A \otimes I/I^2 \xrightarrow{id \otimes \tilde{x}_i} A \otimes k = A$.

3) a) If $\operatorname{Char} k = 0$, then $\varphi : k[X_1 \dots X_n] \to A$ is an isomorphism.
$$X_i \longmapsto x_i$$

b) If $\operatorname{Char} k = p > 0$, then $\varphi : \dfrac{k[X_1 \dots X_n]}{(X_1^p, \dots X_n^p)} \to A$ is an isomorphism.
+ $x_i^p = 0 \ \forall i$,

Proof

- in both cases, φ is surjective by Nakayama
- in both cases, one checks on the x_i's that $\varphi \circ \frac{\partial}{\partial X_i} = D_i \circ \varphi$

$\Rightarrow \ker \varphi$ is stable under $\frac{\partial}{\partial X_i}$.

~~(a)~~ $b \Rightarrow \ker \varphi =$ ~~[illegible]~~ (full ring) or $\ker \varphi = 0$ (since ~~[illegible]~~ $1 \notin \ker \varphi$)

4) Corollary: When $\operatorname{Char} k = 0$, G is trivial. (since $\simeq \mathbb{A}^n$, but $n = 0$)

When $\operatorname{char} k = p$, $\# G$ is a power of p.

($G_1 = \ker F : G \to G^{(p)}$, $G_2 = G/G_1$. previous applies to G_1 and use induction)

5) Corollary: When $\# G$ is invertible, G^0 is trivial, hence G is étale.