Rigid Geometry and Applications

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Talk Plan

- Aim:
  - To give a survey on foundation (based on recent developments).
  - To show how to use it.
- Schedule:
  - I (Mon. 13): What is Rigid Geometry ?
  - II (Tue. 14):
    - Birational Geometry from Zariski's Viewpoint
    - Birational Approach to Rigid Geometry
  - III (Wed. 15): Applications - How to use it.

What is Rigid Geometry ?

$\mathbb{Q}$

$\mathbb{R}$ — $\mathbb{Q}_p$

$\mathbb{C}$ — $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$

$\mathbb{C}$ ⇩ Real - Complex analytic geometry

$\mathbb{C}_p$ ⇩ Rigid analytic geometry

$\mathbb{C}$ vs. $\mathbb{C}_p$

Similarities

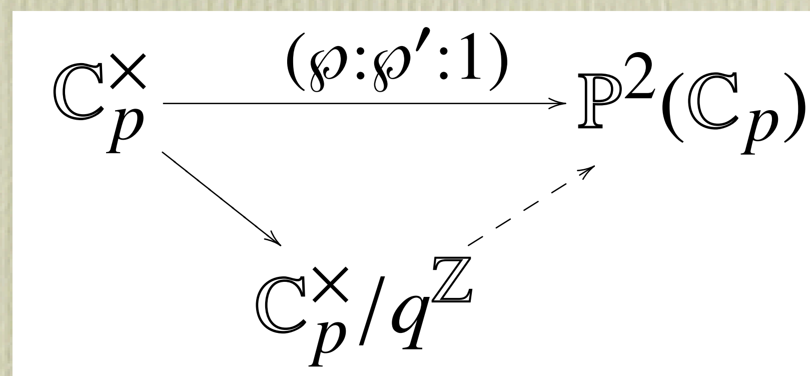
$\mathbb{C}$	$\mathbb{C}_p$
Algebraically closed	Algebraically closed
Complete with respect to the absolute value norm $\lvert\ \rvert_\infty$	Complete with respect to the p-adic norm $\lvert\ \rvert_p$

Analytic Method: Uniformization

$\mathbb{C}$	$\mathbb{C}_p$
$\mathbb{C} \longrightarrow \mathbb{C}/\Lambda$ $\Lambda = \mathbb{Z} + \mathbb{Z} \cdot \tau$ $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$	
Jacobi's uniformization $\mathbb{C}^\times \longrightarrow \mathbb{C}^\times / q^{\mathbb{Z}} = \mathbb{C}/\Lambda$ $q = e^{2\pi\sqrt{-1}\tau}$	Tate's uniformization $\mathbb{C}_p^\times \longrightarrow \mathbb{C}_p^\times / q^{\mathbb{Z}}$ $q \in \mathbb{C}_p^\times, \quad \lvert q\rvert_p < 1$

Tate Curve

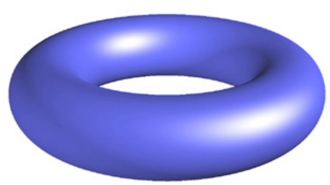
- Embedding onto a cubic curve



- $|j(E)|_p > 1 \iff \exists q \; |q|_p < 1$ such that $E \cong \mathbb{C}_p^\times/q^{\mathbb{Z}}$.

Why do we need analytic methods ?

$E/\mathbb{Q}$

Complex Analysis	Rigid Analysis
$E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ 	$E(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times / q^{\mathbb{Z}}$
	Nagell-Lutz Theorem. K: algebraic number field, E/K: an elliptic curve. $\Longrightarrow E(K)_{\text{tor}}$ is a finite set.

Notation

$(K, |\ |)$: complete non-archimedean valued field with non-trivial valuation $|\ |$.

- $|\ |: K \to \mathbb{R}_{\geq 0}$: *multiplicative* valuation of ht. 1, i.e.,

(1) $|x| = 0 \iff x = 0$.
(2) $|xy| = |x||y|$.
(3) $|x + y| \leq \max\{|x|, |y|\}$.

- $|\ |$: non-trivial $\iff |K^{\times}| \neq \{1\}$.

Tate's Rigid Analytic Geometry

	Algebraic Geometry /k	Rigid Geometry /K
Function algebra	Finitely generated algebra A/k	Topologically finitely generated algebra A/K (called: Affinoid algebra)
Points (Naive)	Maximal ideals of A (with Zariski topology)	Maximal ideals of A (with Admissible topology)
Building Block	Affine variety $(\mathrm{Spm}\, A, \mathcal{O}_X)$	Affinoid $(\mathrm{Spm}\, A, \mathcal{O}_X)$

- There’s no a priori reason why one should take maximal ideals as points.
  (Depends on approaches; there are three others.)
- In Tate’s approach, one has to introduce the admissible topology as a Grothendieck topology.

Example of Affinoid Algebra

$K\langle\langle X_1, \ldots, X_n \rangle\rangle$

$$= \left\{ \begin{array}{r} \sum_{\nu_1,\ldots,\nu_n \geq 0} a_{\nu_1,\ldots,\nu_n} T_1^{\nu_1} \cdots T_n^{\nu_n} \\ \in K[[T_1, \ldots, T_n]] \end{array} \middle| \begin{array}{l} |a_{\nu_1,\ldots,\nu_n}| \to 0 \text{ as} \\ \nu_1 + \cdots + \nu_n \to \infty \end{array} \right\}$$

= The algebra of power series converging absolutely and uniformly on closed unit disk "$\mathbb{D}_K^n$".

Dictionary

Algebraic Geometry $/k = \bar{k}$	Rigid Geometry $/K = \overline{K}$
$k[X_1, \ldots, X_n]$	$K\langle\langle X_1, \ldots, X_n \rangle\rangle$
k^n	$(z_1, \ldots, z_n) \in K^n$ with $\lvert z_i \rvert \leq 1$
$\mathbb{A}^n_k$ Affine space	$\mathbb{D}^n_K$ Closed unit polydisk

Basic Properties

$K\langle\langle X_1, \ldots, X_n\rangle\rangle$:

- Banach algebra with Gauss norm

$$\|\sum_{\nu_1,\ldots,\nu_n\geq 0} a_{\nu_1,\ldots,\nu_n} T_1^{\nu_1}\cdots T_n^{\nu_n}\| = \sup|a_{\nu_1,\ldots,\nu_n}|.$$

- Noetherian and every ideal is closed (w.r.t. the subspace topology).

- Structure of general affinoid algebras:

$$A = K\langle\langle X_1, \ldots, X_n\rangle\rangle / I$$

Banach algebra by the induced norm

Wobbly Topology

- $\operatorname{Spm} A = \{\mathfrak{m} \subset A \mid \text{maximal ideal}\}$.
  - $x \in \operatorname{Spm} A$, $f \in A \rightsquigarrow |f(x)| := |f \bmod x|$.
  - Topology having an open basis

$$\{R(f,g)\}_{f,g \in A}$$

where

$$R(f,g) = \{x \in \operatorname{Spm} A \mid |f(x)| \leq |g(x)|\}.$$

Note: $$R(f,g) = \operatorname{Spm} A\langle\!\langle \tfrac{f}{g} \rangle\!\rangle = \operatorname{Spm} A\langle\!\langle X \rangle\!\rangle/(gX - f).$$

Difficulties

- Spm A is not quasi-compact (w.r.t. the wobbly topology).
- $R(f, g) \mapsto A\langle\langle \frac{f}{g} \rangle\rangle$ is not a sheaf.

$\rightsquigarrow$ Want to "rigidify" the topology:

The name "RIGID" comes from this.

Tate (1961): Realization by using Grothendieck topology.

Admissible Topology

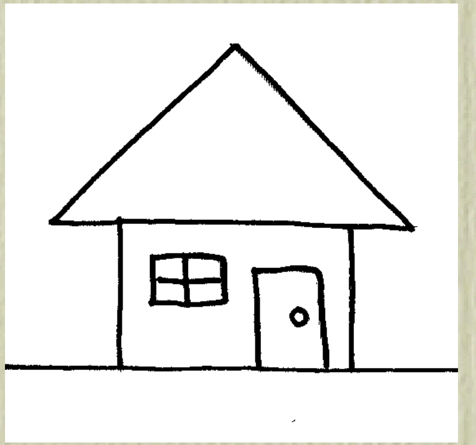
- Grothendieck topology
  - Weaker than wobbly topology.
  - Strongest topology which makes $R(f,g)$ quasi-compact.
- Gives rise to "affinoids" - Building Block.
- General rigid space: By "patching affinoids" w.r.t. admissible topology.

Definition of Admissible Site

- $\mathfrak{A}_K =$ the category of affinoid K-algebras.
- $\{A_i\}_{i\in I}$ (finite collection) covers A

$$\Longleftrightarrow \begin{cases} \text{(a) Each } A_i \text{ is étale over } A. \\ \text{(b) } \operatorname{Spm} A_i \to \operatorname{Spm} A\text{: injective map.} \\ \text{(c) } \operatorname{Spm} A = \bigcup_{i\in I} \operatorname{Spm} A_i. \end{cases}$$

Admissible vs. Wobbly



Admissible land

Wobbly land

Examples.

Annulus

$$\{z \in K \mid |a| \leq |z| \leq |b|\}$$

Affinoid with

Corresponding affinoid algebra

$$= K\langle\langle \tfrac{a}{z}, \tfrac{z}{b} \rangle\rangle$$

$$= K\langle\langle X, Y \rangle\rangle / (XY - \tfrac{a}{b}).$$

Hence, quasi-compact.

Affine line

Realization as the limit of closed disks

$$K = \bigcup_{n \geq 1} \mathbb{D}(0, |a|^{-n}|)$$

$$|a| < 1, \quad \mathbb{D}(0, r) = \{z \in K \mid |z| \leq r\}$$

$$\rightarrow \quad \mathbb{A}_K^{1,\mathrm{an}} = \varinjlim_{n \geq 1} \operatorname{Spm} K\langle\langle a^n z \rangle\rangle$$

Not quasi-compact

Multiplicative group $\mathbb{G}_m$

$$
\begin{aligned}
K^\times &= \bigcup_{n \geq 1} \{z \in K \mid |a|^n \leq |z| \leq |a|^{-n}\} \\
&= \bigcup_{n \in \mathbb{Z}} \{z \in K \mid |a|^{n+1} \leq |z| \leq |a|^n\}
\end{aligned}
$$

$$|a| < 1$$

→ $$\mathbb{G}_{m,K}^{\mathrm{an}} = \bigcup_{n \in \mathbb{Z}} \operatorname{Spm} K\langle\!\langle \frac{a^{n+1}}{z}, \frac{z}{a^n} \rangle\!\rangle$$

Not quasi-compact

Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$ $(|q| < 1)$

Take $a \in K$ with $|a|^k = |q|$ $(k \geq 2)$.

$$\mathbb{G}_{m,K}^{\mathrm{an}} = \bigcup_{n \in \mathbb{Z}} A_n$$

$$A_n = \operatorname{Spm} K\langle\langle \frac{a^{n+1}}{z}, \frac{z}{a^n} \rangle\rangle$$

q maps A_n isomorphically onto A_{n+k}.

$\rightsquigarrow \mathbb{G}_{m,K}^{\mathrm{an}}/q^{\mathbb{Z}}$ as the union of k annuli.

Hence, quasi-compact.

$\mathbb{C}$ vs. $\mathbb{C}_p$ (continued)

Differences

$\mathbb{C}$	$\mathbb{C}_p$
$\nexists$ integer ring	$\exists$ integer ring

Similar for
Affinoid algebras

Can take (non-canonically) “models” of affinoids

Affinoid case

- V: a-adically complete valuation ring of ht. 1
- $K = \mathrm{Frac}(V)$ (with a-adic norm $|\ |$).

A: topologically finitely generated flat V-algebra

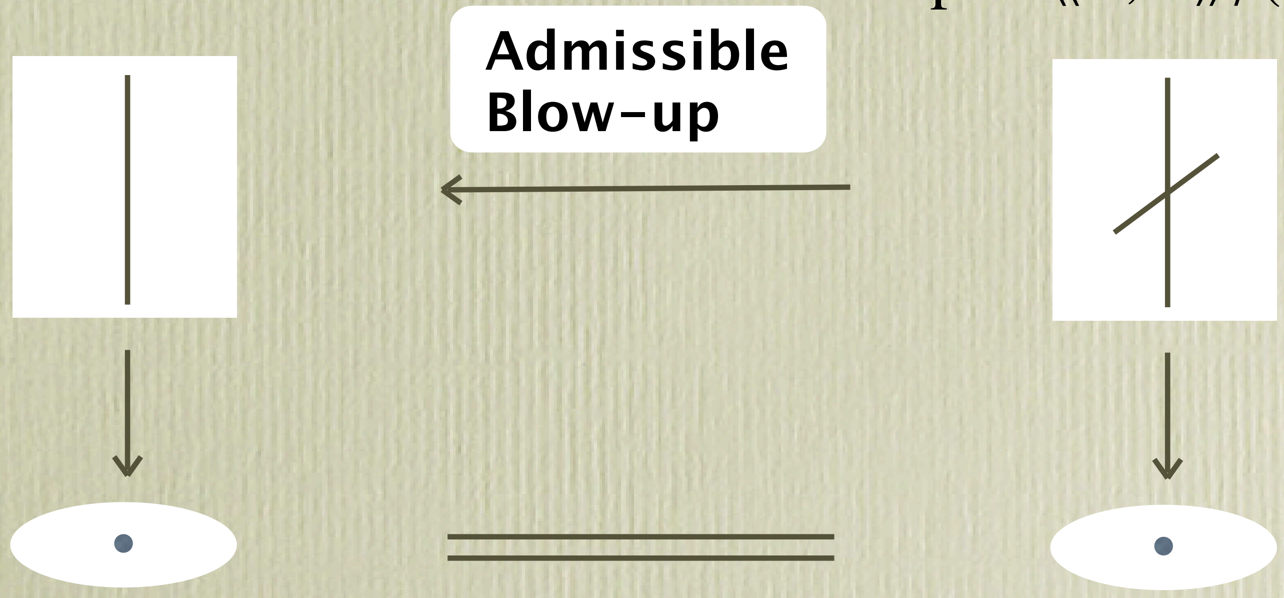
$$\rightsquigarrow A_K = A \otimes_V K\text{: affinoid algebra}/K.$$

Example

Two models of $K\langle\langle X \rangle\rangle$

$\mathrm{Spf}\, V\langle\langle X \rangle\rangle$

$\mathrm{Spf}\, V\langle\langle \frac{X}{a} \rangle\rangle$
$= \mathrm{Spf}\, V\langle\langle X, Y \rangle\rangle / (aY - X)$



$\mathrm{Spf}\, V$

$\mathrm{Spf}\, V$

Zariski Top. vs. Adm. Top.

$$\begin{array}{ccc} U & \hookrightarrow & X \\ & & \downarrow \\ & & \mathrm{Spf}\, A \end{array}$$

Open subset ($U \hookrightarrow X$)

Admissible Blow-up ($X \to \mathrm{Spf}\, A$)

$$\rightsquigarrow \quad U_K \hookrightarrow \mathrm{Spm}\, A_K$$

Open subset w.r.t. admissible topology

Conversely,
Gerritzen-Grauert Theorem $\Longrightarrow$ Enough to recover the admissible topology

Raynaud's viewpoint

Geometry of models $\longrightarrow$ Rigid analytic geometry

$\|$

Geometry of Formal Schemes

Theorems in rigid analytic geometry / DVR $\Longleftarrow$ EGA III

Start from X/V : formal scheme of finite type

$\rightsquigarrow \mathcal{X} = X_K$: rigid analytic space$/K$
(Raynaud Generic fiber)

- Quasi-compact admissible open subset: U_K,

$$U \subseteq X' \overset{\text{adm. blow-up}}{\longrightarrow} X$$

- Point set

$$\begin{aligned} \mathcal{X}(K) &= \{\text{sections } \operatorname{Spf} V \to X\}. \\ U_K(K) &= \{\text{sections which factors through } U\}. \end{aligned}$$

- When $U = \operatorname{Spf} A \rightsquigarrow \Gamma(U_K, \mathcal{O}_{\mathcal{X}}) = A_K$.

Raynaud's Theorem

Coherent formal schemes of finite type / V / Admissible Blow-up $\xrightarrow{\sim}$ Coherent rigid spaces of finite type / K

$$X \longmapsto \mathcal{X} = X_K$$

Raynaud generic fiber

Footnote: Coherent = quasi-compact and quasi-separated

Comments

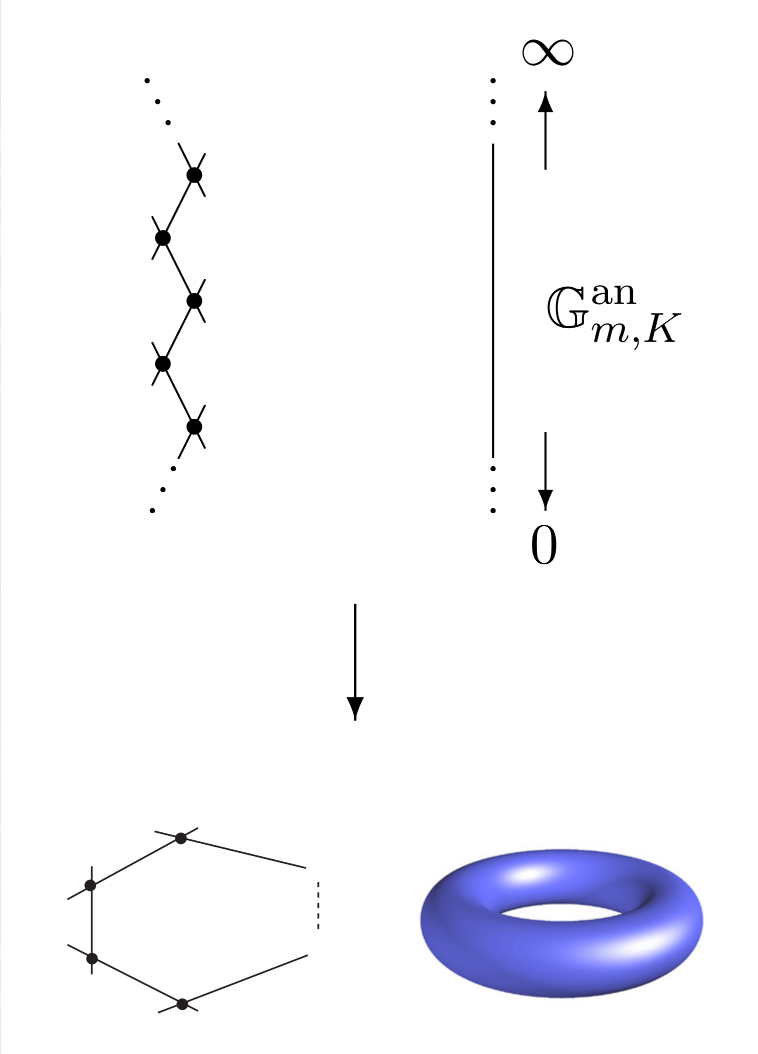
- RHS: defined a priori by "patching affinoids", which turns out to be equivalent to "birational patching" in LHS.
- For the proof:
  - Existence of formal birational patching.
  - Comparing topologies: Gerritzen-Grauert Theorem.
- Significance: Shift from "analysis" to "geometry".

Example: Tate curve

$$\mathbb{G}_{m,K}^{\mathrm{an}} = \bigcup_{n\in\mathbb{Z}} A_n$$

$$\begin{aligned} A_n &= \operatorname{Spm} K\langle\!\langle \tfrac{a^{n+1}}{z}, \tfrac{z}{a^n} \rangle\!\rangle \\ &= \operatorname{Spm} K\langle\!\langle X, Y \rangle\!\rangle/(XY - a) \end{aligned}$$

$$|a|^k = |q| \quad (k \geq 2).$$



Our Approach

General Policy:

Rigid Geometry is a hybrid of formal geometry and birational geometry.

$\rightsquigarrow$

Approach:

Raynaud's approach + Zariski's classical idea