Math 265 Midterm 1

Oct 2nd, 2008

Name: ______________________________

Section: __________

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.
3. You may use one 4-by-6 index card, both sides.

Score		
1	20	
2	20	
3	10	
4	10	
5	20	
6	10	
7	10	
Total	100	

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. A, B, C, X, b are always matrices here.

(a) If $A = A^T$ then A is a square matrix.

(b) If $AB = AC$ and $A \neq 0$ then $B = C$.

(c) Consider a system of linear equations $AX = b$ where A is a square matrix. If the system is inconsistent then $\det(A) = 0$.

(d) Let L be a straight line in the plane $\mathbb{R}^2$. Then L is a subspace of $\mathbb{R}^2$.

(e) If $A^2 = 0$ then A is singular.

	(a)	(b)	(c)	(d)	(e)
Answer	T	F	T	F	T

Explain:

1. If A is a $m \times n$-matrix then A^T is a $n \times m$-matrix. So if $A = A^T$ then $m = n$ and A is a square matrix.
2. Counterexample: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
3. If $\det(A) \neq 0$ then A^{-1} exists and $AX = b$ has a unique solution $X = A^{-1}b$. Thus if $AX = b$ is inconsistent then $\det(A)$ must be 0.
4. A straight line in $\mathbb{R}^2$ is a subspace if and only if the line passing through the origin.
5. $A^2 = 0$ implies that $\det(A)^2 = 0$, hence $\det(A) = 0$ and A is singular.

2. Quick Questions, $A,\ B,\ C,\ X,\ b$ are always matrices here:

(a) Suppose that A is nonsingular and $A^2 = A$. $\det(A) =?$

$\det(A^2) = (\det(A))^2 = \det(A)$. Since A is nonsingular, $\det(A) \neq 0$. So $\det(A) = 1$.

(b) Suppose $AX = 2X$ and $A^{-1}X = aX$. Then $a =?$

Since A exists, times A^{-1} to $AX = 2X$, we get $X = A^{-1}2X$. Hence $A^{-1}X = \frac{1}{2}X$ and $a = \frac{1}{2}$.

(c) Let $\mathbb{R}^{2\times 2}$ be the vector space of 2×2-matrices and $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} |abcd = 0 \right\}$. Is W the subspace of $\mathbb{R}^{2\times 2}$? why?

No. Because it is not closed under addition. For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ are in W. But $A + B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are not in W.

(d) Find a matrix A such that $Y = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}$ is in the null space $N(A)$ of A (Recall that $N(A) = \{X|AX = 0\}$).

We may assume that $A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix}$ a 4×1-matrix. If $Y \in N(A)$ then $AY = 0$. Hence we must have $a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 0 + a_4 \cdot -1 = 0$. Equivalently, $a_1 + 2a_2 - a_4 = 0$. So we may take $a_1 = 1, a_2 = 2, a_3 = 3$ and $a_4 = a_1 + 2a_2 = 5$. Thus $A = (1, 2, 3, 5)$.

3. Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix}.$$

(a) Compute $3A - 2B$.

Solution: $3A - 2B = \begin{pmatrix} 3 & 4 & -5 \\ 4 & 7 & -8 \end{pmatrix}$

(b) Compute AB^T.

Solution:

$$AB^T = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -2 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -4 & -2 \end{pmatrix}$$

4. In the following linear system, determine all values of a for which the resulting linear system has

(a) no solution.

(b) a unique solution

(c) infinitely many solutions

$$\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

Solution:

We have a augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right)$$

Use row operations $r_2 - r_1 \to r_2$ and $r_3 - r_1 \to r_3$, we get

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right)$$

If $a^2 - 4 = 0$ and $a - 2 \neq 0$ then the last leading on locates in the last column, and the system has no solution. Note that $a^2 - 4 = 0$ if and only if $a = \pm 2$, and $a - 2 \neq 0$ if and only if $a \neq 2$. That is, the system has no solution if and only if $a = -2$.

If $a^2 - 4 \neq 0$ then the coefficients matrix is nonsingular and the system has a unique solution. That is if $a \neq \pm 2$ then the system has a unique solution.

Finally, if $a^2 - 4 = a - 2 = 0$ then the system has infinitely many solution. This only happens only $a = 2$.

5. (a) Compute

$$\begin{vmatrix} -2 & 0 & 0 & 0 \\ 5 & 3 & 5 & 7 \\ 3 & 0 & 2 & 1 \\ 8 & 0 & 2 & 2 \end{vmatrix}.$$

Solution: Use cofactor formula for the first row, we have

$$\det(A) = -2(-1)^{1+1} \begin{vmatrix} 3 & 5 & 7 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{vmatrix}.$$

Use cofactor formula for the first column,

$$\det(A) = -2 \cdot 3 \cdot \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = -2 \cdot 3 \cdot 2 = -12.$$

(b) Compute A^{-1}, where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

Solution: Using Gauss-Jordan method, we have $A^{-1} = \begin{pmatrix} -3 & -2 & 2 \\ -4 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}$.

6. Solve the following linear system using Cramer's rule.

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 2 \\ x_1 + 2x_3 &= 0 \\ 2x_1 + 3x_2 - x_3 &= -5 \end{aligned}$$

Solution:

By Cramer's rule, set

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 0 & 2 \\ 2 & 3 & -1 \end{pmatrix}, \ A_1 = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 0 & 2 \\ -5 & 3 & -1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 2 & 2 & 6 \\ 2 & 0 & 2 \\ 2 & -5 & -1 \end{pmatrix} \text{ and } A_3 = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 0 & 0 \\ 2 & 3 & -5 \end{pmatrix}$$

Hence

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-52}{26} = -2.$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{0}{26} = 0.$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{26}{26} = 1.$$

7. Let V be the set of all positive real numbers; define $u \oplus v = uv$ ($\oplus$ is ordinary multiplication) define $\odot$ by $c \odot u = u^c$. Prove that V is a vector space by the following steps:

(a) Verify that V is closed under addition and scalar multiplication.

Let $u, v \in V$ and $c \in \mathbb{R}$. $u \oplus v = uv$ is positive thus $u \oplus v \in V$. $c \odot u = u^c$ is still positive and $c \odot u \in V$. Thus V is closed under addition and scalar multiplication.

(b) Verify $u \oplus v = v \oplus u$ and $(u \oplus v) \oplus w = u \oplus (v \oplus w)$.

$u \oplus v = uv = vu = v \oplus u.$

$(u \oplus v) \oplus w = (uv)w = u(vw) = u \oplus (v \oplus w).$

(c) What is zero vector in the set V, explain?

The zero vector is 1 because for any $u \in V$, $1 \oplus u = 1u = u$.

(d) For any $u \in V$, find $-u$.

$(-u) = \frac{1}{u}$ because $(-u) \oplus u = \frac{1}{u}u = 1$, which is the zero vector proved in last question.

(e) For any real number $c, d \in \mathbb{R}$, and $u, v \in V$, Verify $c \odot (u \oplus v) = c \odot u \oplus c \odot v$; $(c + d) \odot u = c \odot u \oplus d \odot u$; $c \odot (d \odot u) = (c \cdot d) \odot u$ and $1 \odot u = u$.

$$c \odot (u \oplus v) = c \odot (uv) = (uv)^c = u^c v^c = (c \odot u)(c \odot v) = c \odot u \oplus c \odot v.$$

$$(c + d) \odot u = u^{c+d} = u^c u^d = (u^c) \oplus (u^d) = c \odot u \oplus d \odot u$$

$$c \odot (d \odot u) = c \odot (u^d) = (u^d)^c = u^{cd} = (c \cdot d) \odot u$$

$$1 \odot u = u^1 = u.$$